CS 173, Spring 2015
Examlet 3, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

$A = \{(a, b) \in \mathbb{R}^2 \ : \ a = 3 - b^2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

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$A = \{(x, y) \in \mathbb{Z}^2 \mid 2xy + 6y - 5x - 15 \geq 0\}$

$B = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0\}$

$C = \{(p, q) \in \mathbb{Z}^2 \mid q \geq 0\}$

Prove that $(A \cap B) \subseteq C$.

Solution: Suppose that (x, y) is an element of $(A \cap B)$. This means that (x, y) is an element of A and (x, y) is an element of B. So $2xy + 6y - 5x - 15 \geq 0$ and $x \geq 0$, by the definitions of A and B.

Notice that $2xy + 6y - 5x - 15 = (x + 3)(2y - 5)$. So $(x + 3)(2y - 5) \geq 0$. We know that $x + 3$ is positive because $x \geq 0$. So we must have $(2y - 5) \geq 0$.

Now, if $(2y - 5) \geq 0$, then $2y \geq 5$. So $y \geq \frac{5}{2}$. So $y \geq 0$. This means that (x, y) is an element of C which is what we needed to show.

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$A = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$

$B = \{(p, q) \in \mathbb{R}^2 \ : q \geq p^2 - 5\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ |a| \leq 3\}$ (corrected during exam from $C = \{(a, b) \in \mathbb{R}^2 \ : \ |x| \leq 3\}$)

Prove that $A \cap B \subseteq C$.

Solution: It turns out that $A \subseteq C$, so your proof doesn't actually have to involve properties of the set B. [But it's ok if you did them.]

Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $(x, y) \in \mathbb{R}^2$, $x^2 + y^2 \leq 1$.

Notice that x^2 must be non-negative, so $x^2 + y^2 \leq 1$ implies that $x^2 \leq 1$. Therefore $|x| \leq 1$. Therefore $|x| \leq 3$, which is what we needed to show.

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$A = \{(x, y) \in \mathbb{R}^2 \mid x = \lfloor 3y + 5 \rfloor \}$

$B = \{(p, q) \in \mathbb{Z}^2 \mid 2p + q \equiv 3 \pmod 7 \}$

Prove that $A \cap \mathbb{Z}^2 \subseteq B$.

Use the following definition of congruence mod k: if s, t, k are integers, k positive, then $s \equiv t \pmod k$ if and only if $s = t + nk$ for some integer n.

Solution: Let (x, y) be an element of $A \cap \mathbb{Z}^2$. Then (x, y) is an element of A and, also, both x and y are integers.

By the definition of Z, $x = \lfloor 3y + 5 \rfloor$. Since y is an integer, $3y + 5$ must also be an integer. So $\lfloor 3y + 5 \rfloor = 3y + 5$. Therefore, $x = 3y + 5$.

Now, consider $2x + y$.

$$2x + y = 2(3y + 5) + y = 7y + 10 = 7(y + 1) + 3$$

$y + 1$ is an integer, since y is an integer. So this means that $2x + y \equiv 3 \pmod 7$. Therefore, (x, y) is an element of B, which is what we needed to show.

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$A = \{(a, b) \in \mathbb{R}^2 \ : \ b = a^2 - 2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ \lfloor x \rfloor = 4\}$

$C = \{(p, q) \in \mathbb{R}^2 \ : \ 2p \leq q\}$

Prove that $A \cap B \subseteq$ C.

Solution: Let $(p, q) \in \mathbb{R}^2$ and suppose $(p, q) \in A \cap B$. Then $(p, q) \in A$ and $(p, q) \in B$. By the definitions of A and B, this means that $q = p^2 - 2$ and $\lfloor p \rfloor = 4$.

Since $\lfloor p \rfloor = 4$, we know that $4 \leq p < 5$.

Since $p < 5$, $2p < 10$.

Since $p \geq 4$, $q = p^2 - 2 \geq 16 - 2 = 14$.

Therefore $2p < 10 < 14 \leq q$. Since $2p \leq q$, $(p, q) \in C$, which is what we needed to show.

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$A = \{(x, y) \in \mathbb{R}^2 \;:\; 0.3 \leq xy \leq 10.5\}$

$B = \{(a, b) \in \mathbb{Z}^2 \;:\; a > 4\}$

$C = \{(p, q) \in \mathbb{Z}^2 \;:\; q \leq 2\}$

Prove that $A \cap B \subseteq$ C.

Solution: Let $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$. By the definitions of A and B, this means that $(x, y) \in mathbbZ^2$, so x and y are integers. Also $0.3 \leq xy \leq 10.5$ and $x > 4$.

Since x and y are integers, $0.3 \leq xy \leq 10.5$ implies that $1 \leq xy \leq 10$. Also, since x is an integer, $x > 4$ implies that $x \geq 5$.

Since $x \geq 5$ and y is positive, $xy \geq 5y$. Since we also know that $xy \leq 10$, we have $10 \geq 5y$. So $y \leq 2$

Since (x, y) is a pair of integers with $y \leq 2$, (x, y) is an element of C by the definition of the set C. This is what we needed to prove.