CS 173, Spring 2015
Examlet 4, Part A

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Discussion: Monday 9 10 11 12 1 2 3 4 5

Define the relation $\sim$ on $\mathbb{Z}$ by

$x \sim y$ if and only if $5 \mid (3x + 7y)$

Working directly from the definition of divides, prove that $\sim$ is transitive.

Solution: Let x, y, and z be integers. Suppose that $x \sim y$ and $y \sim z$.

By the definition of $\sim$, $5 \mid (3x + 7y)$ and $5 \mid (3y + 7z)$. So $3x + 7y = 5m$ and $3y + 7z = 5n$, for some integers m and n.

Adding these two equations together, we get $3x+7y+3y+7z = 5m+5n$. So $3x+10y+7z = 5(m+n)$. So $3x + 7z = 5(m + n - 2y)$.

$m + n - 2y$ is an integer, since m, n and y are integers. So this means that $5 \mid 3x + 7z$ and therefore $x \sim z$, which is what we needed to show.

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $(xy)(p + q) = (pq)(x + y)$

Prove that T is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $(xy)(p + q) = (pq)(x + y)$ and $(pq)(m + n) = (mn)(p + q)$

Since $m+n$ is positive, we can divide both sides by it, to get $(pq) = (mn)(p+q)/(m+n)$. Substituting this into the first equation, we get

$$(xy)(p + q) = (mn)(p + q)/(m + n) \times (x + y)$$

Multiplying both sides by $(m + n)$, we get

$$(xy)(p + q)(m + n) = (mn)(p + q)(x + y)$$

Since $(p + q)$ is positive, we can cancel it from both sides to get

$$(xy)(m + n) = (mn)(x + y)$$

By the definition of T, this means that $(a, b)T(m, n)$, which is what we needed to show.

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Recall how to multiply a real number α by a 2D point $(x, y) \in \mathbb{R}^2$: $\alpha(x, y) = (\alpha x, \alpha y)$.

Let $A = \mathbb{R}^+ \times \mathbb{R}^+$, i.e. pairs of positive real numbers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists a real number $\alpha \geq 1$ such that $(x, y) = \alpha(p, q)$.

Prove that $\gg$ is antisymmetric.

Solution: Let (x, y) and (p, q) be elements of A. Suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (x, y)$.

By the definition of $\gg$, there are real numbers $\alpha \geq 1$ and $\beta \geq 1$ such that $(x, y) = \alpha(p, q)$ and $(p, q) = \beta(a, b)$.

Substituting the second equation into the first, we get $(x, y) = \alpha\beta(x, y)$. This means that $\alpha\beta = 1$. Since $\alpha \geq 1$ and $\beta \geq 1$, this implies that $\alpha = \beta = 1$. So therefore $(x, y) = (p, q)$, which is what we needed to show.

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $x \leq p$ and $xy \leq pq$

Prove that T is antisymmetric.

Solution: Let (x, y) and (p, q) be elements of A. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$.

By the definition of T, $(x, y)T(p, q)$ implies that $x \leq p$ and $xy \leq pq$.

Similarly $(p, q)T(x, y)$ implies that that $p \leq x$ and $pq \leq xy$.

Since $x \leq p$ and $p \leq x$, $x = p$. Since $xy \leq pq$ and $pq \leq xy$, $xy = pq$.

Notice that x and o are positive, by the definition of A. So $x = p$ and $xy = pq$ implies that $y = q$.

We now know that $x = p$ and $y = q$. So therefore $(x, y) = (p, q)$, which is what we needed to show.

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Let $A = \mathbb{N} \times \mathbb{N}$, i.e. pairs of natural numbers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists an integer $n \geq 1$ such that $(x, y) = (np, nq)$.

Prove that $\gg$ is transitive.

Solution: Let (x, y), (p, q) and (a, b) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (a, b)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(a, b)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = ma$ and $q = mb$.

Combining these equations, we get $x = np = n(ma) = (nm)a$ and $y = nq = n(mb) = (nm)b$. Let $s = nm$. Since m and n are positive integers, so is s. But $(x, y) = (sa, sb)$. So $(x, y) \gg (a, b)$, which is what we needed to show.

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A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$$(c, r) \preceq (d, q) \text{ if and only if } r \leq q \text{ and } |c - d| + r \leq q.$$

Prove that $\preceq$ is antisymmetric.

Solution: Let (c, r) and (d, q) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (c, r)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $r \leq q$ and $|c - d| + r \leq q$. Similarly, $(d, q) \preceq (c, r)$ means that $q \leq r$ and $|d - c| + q \leq r$.

Since $r \leq q$ and $q \leq r$, $q = r$. Substituting this into $|c - d| + r \leq q$, we get $|c - d| + r \leq r$. So $|c - d| \leq 0$. Since the absolute value of a real number cannot be negative, this means that $|c - d| = 0$, so $c = d$.

Since $q = r$ and $c = d$, $(c, r) = (d, q)$, which is what we needed to prove.