CS 173, Spring 2015
Examlet 5, Part A

NETID:

FIRST: | **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) - y, 5y + 3)$. Prove that g is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of integers and suppose that $g(x, y) = g(a, b)$.

   By the definition of g, we know that $f(x) - y = f(a) - b$ and $5y + 3 = 5b + 3$. Since $5y + 3 = 5b + 3$, $5y = 5b$, so $y = b$. Substituting this back into the $f(x) - y = f(a) - b$, we find $f(x) - y = f(a) - y$, so $f(x) = f(a)$. Since f is one-to-one, this implies that $x = a$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

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1. (10 points) If a is any real number, (a, ∞) is the set of all real numbers greater than a. Let's define the function $f : (0, \infty) \to \left(\frac{1}{3}, \infty\right)$ by $f(x) = \dfrac{x^2 + 2}{3x^2}$. Prove that f is onto.

   Solution: Let $y \in \left(\frac{1}{3}, \infty\right)$. Then $y > \frac{1}{3}$, so $3y > 1$, and therefore $3y - 1 > 0$.

   So $\frac{2}{3y-1}$ is defined and positive. So consider $x = \sqrt{\frac{2}{3y-1}}$. x is defined and belongs to $(0, \infty)$.

   Then $x^2 = \frac{2}{3y-1}$. So $x^2 + 2 = \frac{2}{3y-1} + 2 = \frac{2+(6y-2)}{3y-1} = \frac{6y}{3y-1}$. And $3x^2 = \frac{6}{3y-1}$.

   Then $f(x) = \frac{x^2+2}{3x^2} = \frac{6y}{6} = y$.

   So we've found a pre-image for our original value y, which is what we needed to do.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that $h : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $f(x, y) = (h(x) - y, 3h(x) + 1)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(h(x) - y, 3h(x) + 1) = (h(p) - q, 3h(p) + 1)$. So $h(x) - y = h(p) - q$ and $3h(x) + 1 = 3h(p) + 1$.

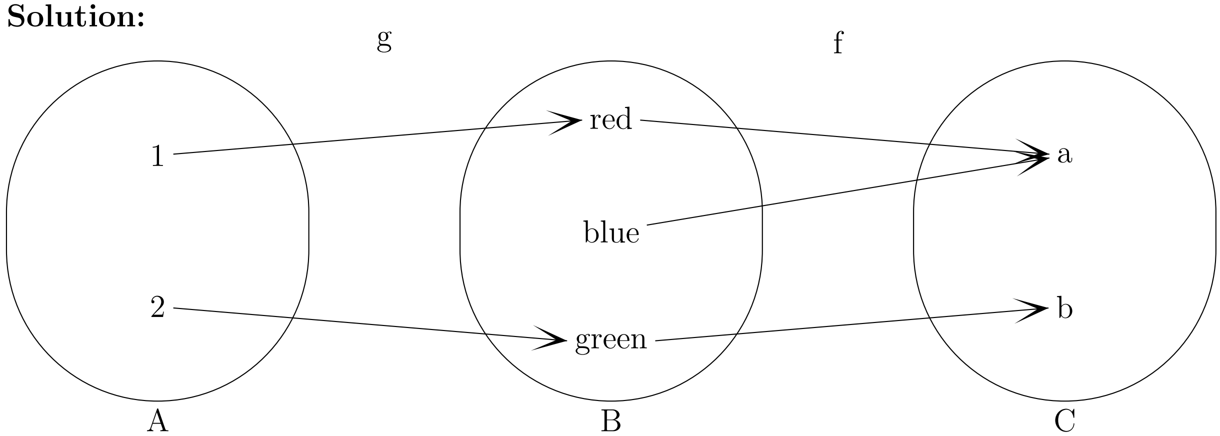
   Since $3h(x) + 1 = 3h(p) + 1$, $3h(x) = 3h(p)$. So $h(x) = h(p)$. Since h is one-to-one, this means that $x = p$.

   But, also, since $h(x) = h(p)$ and $h(x) - y = h(p) - q$, we know that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q0$, which is what we needed to prove.

2. (5 points) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is one-to-one, then f is one-to-one. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

   Solution:



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1. (10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

   By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

   Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

   Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in C, there is an element x in M such that $g(x) = y$.

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1. (10 points) Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and let $f : \mathbb{N}^2 \to \mathbb{Z}$ be defined by

$$f(n, m) = (m - 1)g(n)$$

Prove that f is onto.

Solution: Let a be an integer.

Case 1) $a \geq 0$. Since g is onto, we can find a natural number n such that $g(n) = a$. Let $m = 2$. Then $f(n, m) = (2 - 1)g(n) = 1 \cdot a = a$.

Case 2) $a \leq 0$. Then $(-a)$ is a natural number. Since g is onto, we can find a natural number n such that $g(n) = (-a)$. Let $m = 0$. Then $f(n, m) = (0 - 1)g(n) = (-1) \cdot (-a) = a$.

So we've found a point (n, m) such that $g(n, m) = a$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $h : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $h(x, y) = (g(x) + g(y), g(x) - g(y))$. Prove that h is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $h(x, y) = h(p, q)$.

   By the definition of h, this means that $(g(x) + g(y), g(x) - g(y)) = (g(p) + g(q), g(p) - g(q))$. So $g(x) + g(y) = g(p) + g(q)$ and $g(x) - g(y) = g(p) - g(q)$.

   Adding these equations together, we get $2g(x) = 2g(p)$. So $g(x) = g(p)$. Since g is one-to-one, this implies that $x = p$.

   Similarly, if we subtract the two equations, we get $2g(y) = 2g(q)$. So $g(y) = g(q)$. And since g is one-to-one, $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.