CS 173, Spring 2015
Examlet 7, Part A

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Discussion: Monday 9 10 11 12 1 2 3 4 5

Use (strong) induction to prove the following claim:

Claim: For all integers a, b, n, $n \geq 1$, if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$.

Use this definition in your proof: $x \equiv y \pmod p$ if and only if $x = y + kp$ for some integer k.

Proof by induction on n.

Base case(s): At $n = 1$, our claim becomes "if $a \equiv b \pmod 7$ then $a \equiv b \pmod 7$" which is clearly true.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for all integers a, b, n, where $n = 1, \ldots, k$,

a and b need to be introduced at some point in this proof, but there's several places you might do this. For example, you could say "let a and b be integers" right at the start. Then your inductive hypothesis would just be "if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for $n = 1, \ldots, k$." We won't get picky about this when grading.

Rest of the inductive step:

Let a and b be integers.

Suppose that $a \equiv b \pmod 7$. then $a = b + 7p$ for some integer p.

From the inductive hypothesis, we know that $a^k \equiv b^k \pmod 7$, So $a^k = b^k + 7q$ for some integer q.

Combining these two equations, we get that

$$a^{k+1} = (b + 7p)(b^k + 7q) = b^{k+1} + 7(pb^k + bq + 7pq)$$

$pb^k + bq + 7pq$ is an integer since p, q, and b are integers. So we know that $a^{k+1} \equiv b^{k+1} \pmod 7$, which is what we needed to prove.

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Use (strong) induction to prove the following claim

Claim: $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, for all natural numbers n and all real numbers $p \neq 1$.

Proof by induction on n.

Base case(s): at $n = 0$, $\sum_{k=0}^{n} p^k = p^0 = 1$. And $\frac{p^{n+1}-1}{p-1} = \frac{p-1}{p-1} = 1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, all real numbers $p \neq 1$. and all natural numbers $n = 0, \ldots, j$.

p needs to be introduced somewhere, but there are several options. For example, you could say "let p be a real number $\neq 1$" before you give the inductive hypothesis. We won't be picky about this when grading.

Notice that the moving variable for the summation is k, so you can't also use k for the bound on the induction variable. You need to use a fresh variable name for one of the two.

Rest of the inductive step: Let p be a real number $\neq 1$.

Then $\sum_{k=0}^{j+1} p^k = p^{j+1} + \sum_{k=0}^{j} p^k$

By the inductive hypothesis, we know that $\sum_{k=0}^{j} p^k == \frac{p^{j+1}-1}{p-1}$. Substituting this into the previous equation, we get

$$\sum_{k=0}^{j+1} p^k = p^{j+1} + \frac{p^{j+1}-1}{p-1} = \frac{p^{j+1}(p-1) + p^{j+1} - 1}{p-1} = \frac{p^{j+2} - p^{j+1} + p^{j+1} - 1}{p-1} = \frac{p^{j+2}-1}{p-1}$$

So $\sum_{k=0}^{j+1} p^k = \frac{p^{j+2}-1}{p-1}$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for all positive integers n.

Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} j(j+1) = 1(1+1) = 2$ Also, $\frac{n(n+1)(n+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$. So the two sides of the equation are equal at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for $n = 1, \ldots, k$, for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} j(j+1)$. By removing the top term of the summation and applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} j(j+1) = (k+1)(k+2) + \sum_{j=1}^{k} j(j+1) = (k+1)(k+2) + \frac{k(k+1)(k+2)}{3}$$

Simplifying the algebra:

$(k+1)(k+2) + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)}{3} + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2) + k(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$

So $\sum_{j=1}^{k+1} j(j+1) = \frac{(k+1)(k+2)(k+3)}{3}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for all positive integers n.

Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$. Also $\frac{n}{n+1} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for $n = 1, \ldots, k$ for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} \frac{1}{j(j+1)}$. By removing the top term of the summation and then applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \frac{k}{k+1}.$$

Adding the two fractions together:

$$\frac{1}{(k+1)(k+2)} + \frac{k}{k+1} = \frac{1}{(k+1)(k+2)} + \frac{k(k+2)}{(k+1)(k+2)} = \frac{1+k(k+2)}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

So $\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{k+2}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $2^{n+2} + 3^{2n+1}$ is divisible by 7, for all natural numbers n.

Proof by induction on n.

Base case(s): At $n = 0$, $2^{n+2} + 3^{2n+1} = 2^2 + 3 = 7$ which is clearly divisible by 7.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $2^{n+2} + 3^{2n+1}$ is divisible by 7, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

At $n = k + 1$, $2^{n+2} + 3^{2n+1}$ is equal to $2^{k+3} + 3^{2k+3}$.

$2^{k+3} + 3^{2k+3} = 2 \cdot 2^{k+2} + 9 \cdot 3^{2k+1} = 2(2^{k+2} + 3^{2k+1}) + 7(3^{3k+1})$

By the inductive hypothesis, $2^{k+2}+3^{2k+1}$ is divisible by 7. So $2(2^{k+2}+3^{2k+1})$ is divisible by 7. $7(3^{3k+1})$ is divisible by 7 because 3^{3k+1} is an integer. So the sum of these two terms must be divisible by 7.

Thus, $2^{k+3} + 3^{2k+3}$ is divisible by 7, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for all positive integers n

Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} 2(-1)^p p^2 = 2(-1)^1 1^2 = -2$. And $(-1)^n n(n+1) = (-1)^1 1 \cdot 2 = -2$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for $n = 1, 2, \ldots k$.

Rest of the inductive step:

$$\sum_{p=1}^{k+1} 2(-1)^p p^2 = 2(-1)^{k+1}(k+1)^2 + \sum_{p=1}^{k} 2(-1)^p p^2$$

By the inductive hypothesis, we know that $\sum_{p=1}^{k} 2(-1)^p p^2 = (-1)^k k(k+1)$. Substituting this into the previous equation, we get

$$\begin{aligned}
\sum_{p=1}^{k+1} 2(-1)^p p^2 &= 2(-1)^{k+1}(k+1)^2 + (-1)^k k(k+1) \\
&= (k+1)(-1)^{k+1}(2(k+1) - k) \\
&= (k+1)(-1)^{k+1}(k+2) = (-1)^{k+1}(k+1)(k+2)
\end{aligned}$$

So $\sum_{p=1}^{k+1} 2(-1)^p p^2 = (-1)^{k+1}(k+1)(k+2)$ which is what we needed to show.