CS 173, Spring 2015
Examlet 8, Part A

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Discussion: **Monday** **9** **10** **11** **12** **1** **2** **3** **4** **5**

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 3 \qquad f(2) = 5$

$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 3$.

Use induction to prove that $f(n) = 2^n + 1$

Proof by induction on n.

Base case(s):

Solution:

$n = 1$: $f(1) = 3$. Also $2^1 + 1 = 3$.

$n = 2$: $f(2) = 5$. Also $2^2 + 1 = 5$.

So the claim holds for both $n = 1$ and $n = 2$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = 2^n + 1$ for $n = 1, 2, \ldots, k-1$.

Rest of the inductive step:

Solution: By the definition of f and the inducive hypothesis, we get that

$$\begin{aligned} f(k) &= 3f(k-1) = 2f(k-2) \\ &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) \end{aligned}$$

Simplifying the algebra, we get:

$$\begin{aligned} f(k) &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} - 2 \\ &= (3-1)2^{k-1} + (3-2) = 2^k + 1 \end{aligned}$$

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(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$f(0) = f(1) = f(2) = 1$$

$$f(n) = f(n-1) + f(n-3), \text{ for all } n \geq 3$$

Use induction to prove that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$ You may use the fact that $\sqrt{2}$ is smaller than 1.5.

Proof by induction on n.

Base case(s): For $n = 0$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}$. For $n = 1$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2}) = \frac{1}{\sqrt{2}}$. For $n = 2$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2}(2) = 1$ In all three cases, the value is $\leq 1 = f(n)$.

Solution:

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step:

Solution:

Using the definition of f and the inductive hypothesis, we get

$$f(k) = f(k-1) + f(k-3) \geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3}$$

Simplifying this expression, we get

$$\begin{aligned} f(k) &\geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3} = \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}\frac{1}{2}(\sqrt{2})^{k-1} \\ &= \frac{1}{2}(\sqrt{2})^{k-1}(1 + \frac{1}{2}) = \frac{1}{2}(\sqrt{2})^{k-1}(1.5) \\ &\geq \frac{1}{2}(\sqrt{2})^{k-1}(\sqrt{2}) = \frac{1}{2}(\sqrt{2})^k \end{aligned}$$

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5 \qquad f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Proof by induction on n.

Base case(s):

Solution: For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.
For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:

Solution:

Using the definition of f and the inductive hypothesis, we get

$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$

Simplifying the algebra,

$$\begin{aligned} 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\ &= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\ &= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3 \end{aligned}$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by:

$f(1) = 3 \qquad f(2) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for all $n \geq 3$

Use induction to prove that $f(n) \leq 3^n$

Proof by induction on n.

Base case(s):

Solution:

For $n = 1$, $f(n) = 3$ and $3^n = 3$, so $f(n) \leq 3^n$.

For $n = 2$, $f(n) = 7$ and $3^n = 3^2 = 9$, so $f(n) \leq 3^n$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $f(n) \leq 3^n$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$.

Rest of the inductive step:

Solution:

By the inductive hypothesis, we know that $f(k-1) \leq 3^{k-1}$ and $f(k-2) \leq 3^{k-2}$. So, using these two inequalities plus the definition of f, we get:

$f(k) = f(k-1) + 2f(k-2) \leq 3^{k-1} + 2 \cdot 3^{k-2}$

But then

$3^{k-1} + 2 \cdot 3^{k-2} \leq 3^{k-1} + 2 \cdot 3^{k-1} = 3 \cdot 3^{k-1} = 3^k$

So $f(k) \leq 3^k$, which is what we needed to show.

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(20 points) Suppose that $P : \mathbb{N} \to \mathbb{N}$ is defined by

$P(0) = 2 \qquad P(1) = 1$

$P(n) = P(n-1) + 6P(n-2)$, for all $n \geq 2$

Use induction to prove that $P(n) = 3^n + (-2)^n$

Proof by induction on n.

Base case(s):

Solution:

$n = 0$: $P(0) = 2$. Also $3^n + (-2)^n = 3^0 + (-2)^0 = 1 + 1 = 2$. So the claim holds at $n = 0$.

$n = 1$: $P(1) = 1$. Also $3^n + (-2)^n = 3^1 + (-2)^1 = 3 - 2 = 1$. So the claim holds at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $P(n) = 3^n + (-2)^n$ for $n = 0, 1, \ldots, k-1$, for some integer $k \geq 2$.

Rest of the inductive step:

Solution:

$$\begin{aligned}
P(k) &= P(k-1) + 6P(k-2) \quad \text{by the definition of P} \\
&= (3^{k-1} + (-2)^{k-1}) + 6(3^{k-2} + (-2)^{k-2}) \quad \text{by the inductive hypothesis} \\
&= 3^{k-1} + (-2)^{k-1} + 6 \cdot 3^{k-2} + 6 \cdot (-2)^{k-2} \\
&= 3^{k-1} + (-2)^{k-1} + 2 \cdot 3^{k-1} - 3 \cdot (-2)^{k-1} \\
&= 3 \cdot 3^{k-1} - 2 \cdot (-2)^{k-1} \\
&= 3^k + (-2)^k
\end{aligned}$$

So $P(k) = 3^k + (-2)^k$, which is what we needed to show.

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(20 points) Suppose that $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is defined by

$f(1) = 0 \qquad f(2) = 12$

$f(n) = 4f(n-1) - 3f(n-2), \quad \text{for } n \geq 3$

Use induction to prove that $f(n) = 2 \cdot 3^n - 6$

Proof by induction on n.

Base case(s):

Solution: For $n = 1$, $f(1) = 0$ and $2 \cdot 3^n - 6 = 2\dot{3} - 6 = 0$. So the claim is true.

For $n = 2$, $f(2) = 12$ and $2 \cdot 3^n - 6 = 2\dot{3}^2 - 6 = 18 - 6 = 12$. So the claim is true.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = 2 \cdot 3^n - 6$ for $n = 1, 2, \ldots, k-1$ for some positive integer $k \geq 3$.

Rest of the inductive step:

Solution: $f(k) = 4 \cdot f(k-1) - 3 \cdot f(k-2)$ by the definition of f.

So $f(k) = 4 \cdot (2 \cdot 3^{k-1} - 6) - 3 \cdot (2 \cdot 3^{k-2} - 6)$ by the inductive hypothesis.

So $f(k) = 8 \cdot 3^{k-1} - 24 - 6 \cdot 3^{k-2} + 18 = 8 \cdot 3^{k-1} - 2 \cdot 3^{k-1} - 6 = 6 \cdot 3^{k-1} - 6 = 2 \cdot 3^k - 6$

So $f(k) = 2 \cdot 3^k - 6$ which is what we needed to show.