CS 173, Spring 2015
Examlet 8, Part B

NETID:

FIRST: | **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= 17 \\ F(n) &= 3F(n/2), \text{ for } n \geq 4 \end{aligned}$$

Use unrolling to find the closed form for F. Show your work and simplify your answer.

Solution:

$$\begin{aligned} F(n) &= 3F(n/2) = 3(3F(n/4)) = 3(3(3(F(n/2^3)))) \\ &= 3^3F(n/2^3) \\ &= 3^kF(n/2^k) \end{aligned}$$

We'll hit the base case when $n/2^k = 2$, i.e. $n = 2^{k+1}$, i.e. $k+1 = \log_2 n$, $k = \log_2 n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} F(n) &= 3^{\log_2 n - 1} \cdot 17 \\ &= 17/3 \cdot 3^{\log_2 n} = 17/3 \cdot 3^{\log_3 n \log_2 3} \\ &= 17/3n^{\log_2 3} \end{aligned}$$

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1. (8 points) Suppose we have a function g defined by

$$\begin{aligned} g(0) &= g(1) = c \\ g(n) &= kg(n-2) + n^2, \text{ for } n \geq 2 \end{aligned}$$

where k and c are constants. Express $g(n)$ in terms of $g(n-6)$ (where $n \geq 6$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} g(n) &= kg(n-2) + n^2 \\ &= k(kg(n-4) + (n-2)^2) + n^2 \\ &= k(k(kg(n-6) + (n-4)^2) + (n-2)^2) + n^2 \\ &= k^3 g(n-6) + k^2(n-4)^2 + k(n-2)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The number of edges in the 4-dimensional hypercube Q_4 — 5 ☐ 12 ☐ 32 ☑ 64 ☐

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1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} f(n) &= 3f(n-1) + n^2 \\ &= 3(3f(n-2) + (n-1)^2) + n^2 \\ &= 3(3(3f(n-3) + (n-2)^2) + (n-1)^2) + n^2 \\ &= 27f(n-3) + 9(n-2)^2 + 3(n-1)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The diameter of the 4-dimensional hypercube Q_4 1 ☐ 2 ☐ 4 ☑ 16 ☐

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(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= c \\ F(n) &= F(n/2) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$F(n) = F(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i}$$

Finish finding the closed form for F. Show your work and simplify your answer.

Solution:

To find the value of k at the base case, we need to set $n/2^k = 2$. This means that $n = 2 \cdot 2^k$. So $n = 2^{k+1}$. So $k + 1 = \log n$. So $k = \log n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} T(n) &= T(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i} = T(4) + \sum_{i=0}^{\log n-2} n\frac{1}{2^i} \\ &= c + n\sum_{i=0}^{\log n-2} \frac{1}{2^i} = c + n(2 - \frac{1}{2^{\log n-2}}) \\ &= c + n(2 - \frac{1}{2^{\log n} \cdot 2^{-2}}) = c + n(2 - \frac{4}{2^{\log n}}) \\ &= c + n(2 - \frac{4}{n}) = c + 2n - 4 \end{aligned}$$

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1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 3 \end{aligned}$$

Express $g(n)$ in terms of $g(n/3^3)$ (where $n \geq 27$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} g(n) &= 3g(n/3) + n \\ &= 3(3g(n/9) + n/3) + n \\ &= 3(3(3g(n/27) + n/9) + n/3) + n \\ &= 27g(n/27) + n + n + n \\ &= 27g(n/27) + 3n \end{aligned}$$

2. (2 points) Define the Fibonacci numbers

Solution:

$F_0 = 0 \qquad F_1 = 1$

or $F_1 = F_2 = 1$

$F_n = F_{n-1} + Fn - 2$

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(10 points) Suppose we have a function F defined (for n a power of 3) by

$$\begin{aligned} F(1) &= 5 \\ F(n) &= 3F(n/3) + 7 \text{ for } n \geq 3 \end{aligned}$$

Your partner has already figured out that

$$F(n) = 3^k F(n/3^k) + 7\sum_{p=0}^{k-1} 3^p$$

Finish finding the closed form for F. Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

Solution:

To find the value of k at the base case, set $n/3^k = 1$. Then $n = 3^k$, so $k = log_3 n$. Substituting this into the above, we get

$$\begin{aligned} F(n) &= 3^{\log_3 n} \cdot 5 + 7 \sum_{p=0}^{\log_3 n - 1} 3^p \\ &= 5n + 7\frac{3^{log_3 n} - 1}{3-1} \\ &= 5n + 7\frac{n-1}{3-1} = 5n + \frac{7(n-1)}{2} \end{aligned}$$