CS 173, Spring 2015
Examlet 9, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(18 points) Here is a grammar G, with start symbol S and terminal symbols a and b.

$$\begin{aligned} S &\rightarrow a\ S\ a \mid S\ a\ S \mid a\ N\ a \\ N &\rightarrow a \mid b\ b \end{aligned}$$

Use (strong) induction to prove that any tree of height h matching (aka generated by) grammar G has at least h nodes with label a. Use $A(T)$ as shorthand for the number of a's in a tree T.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest trees generated by G have $h = 2$. They are as shown below and, as you can see, they both have at least two nodes labelled a.

```
    S              S
  / | \          / | \
 a  N  a        a  N  a
    |             / \
    a            b   b
```

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: All trees of height h generated by G have at least h nodes labelled a, for $h = 2, 3, \ldots, k-1$. ($k \geq 3$)

Inductive Step: Suppose that T is a tree generated by G of height k. There are two cases:

Case 1: T consists of a root labelled S, with three children. The left and right children have label a. The middle child is a subtree T_1 whose root has label S. T_1 must have height $k-1$ so, by the inductive hypothesis, it contains at least $k-1$ a's. So T contains at least $(k-1)+2 = k+1$ a's.

Case 1: T consists of a root labelled S, with three children. The middle child child has label a. The left and right children are subtrees T_1 and T_2 whose roots have label S. At least one of these two subtrees has height $k-1$ so, by the indutive hypothesis, it contains at least $k-1$ a's. The middle child of T adds another a, so T must have at least k a's. (The other subtree may add additional a's.)

In either case, T has at least k nodes labelled a, which is what we needed to show.

CS 173, Spring 2015 Examlet 9, Part A

NETID:

FIRST:	**LAST:**

Discussion: **Monday 9 10 11 12 1 2 3 4 5**

(18 points) Let's define a Filbert tree to be a binary tree containing 2D points such that:

- Each leaf node contains $(3,1)$, $(-2,-5)$, or $(2,2)$.
- An internal node with one child labelled (a,b) has label $(a+1,b-1)$.
- An internal node with two childen labelled (x,y) and (a,b) has label $(x+a,y+b)$.

Use (strong) induction to prove that the point in the root node of any Filbert tree is on or below the line $x=y$.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest Filbert trees consist of a single node containing $(3,1)$, $(-2,-5)$, or $(2,2)$. All three of these points lie on or below the line $x=y$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the point in the root node of any Filbert tree is on or below the line $x=y$, for trees of height $h=0,1,\ldots,k-1$. $(k \geq 1)$.

Inductive Step: Let T be a Filbert tree of height k. There are two cases.

Case 1: The root of T has one child subtree, whose root contains (a,b). By the inductive hypothesis, (a,b) is on or below $x=y$, i.e. $b \leq a$. By the definition of a Filbert tree, the root of T contains $(a+1,b-1)$. Since $b \leq a$, $b-1 \leq a+1$, so this point is on or below $x=y$.

Case 2: The root of T has two child subtrees, whose roots contain (x,y) and (a,b). Then the root of T contains $(x+a,y+b)$. By the inductive hypothesis, $y \leq x$ and $b \leq a$, so $y+b \leq x+a$. So $(x+a,y+b)$ is on or below $x=y$.

In both cases the root note contains a point on or below $x=y$, which is what we needed to show.

CS 173, Spring 2015 Examlet 9, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

Recall that a full binary tree is one in which every node has either 0 or 2 children. An *Illini tree* is a full binary tree in which each node is colored orange or blue, such that:

- If v is a leaf node, then v may be colored orange or blue.
- If v has two children of the same color, then v is colored blue.
- If v has two children of different colors, then v is colored orange.

Use (strong) induction to show that the root of an Illini tree is blue if and only if the tree has an even number of orange leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): An Illini trees with $H = 0$ consists of a single node. If it's blue, the tree contains no orange nodes, which is even. If it's orange, the tree contains one orange node, which is odd. In both cases, the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the root of an Illini tree is blue if and only if the tree has an even number of orange leaves, for trees of height $h = 0, 1, \ldots, k-1$. $(k \geq 1)$

Inductive Step: Let T be an Illini tree of height k. There are two cases:

Case 1: the root of T is colored blue and it has two child subtrees whose roots are the same color. If both are blue, then both subtrees contain an even numbr of orange leaves by the inductive hypothesis. Similarly, if both are orange, then each contains an odd number of orange leaves. Since two odd numbers, or two even numbers, sum to an even number, T has an even number of orange leaves.

Case 2: the root of T is colored orange and it has two child subtrees whose roots are opposite colors. By the inductive hypothesis, the subtree with an orange root contain an odd number of orange leaves and the subtree with a blue root contains an even number of orange leaves. So T contains an odd number of orange leaves.

In both cases, T contains an even number of orange leaves if and only if its root is blue, which is what we needed to show.

CS 173, Spring 2015 Examlet 9, Part A

NETID:

FIRST: | **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(18 points) Let's define an Interp Tree to be a binary tree containing 2D points such that:

- Each leaf node contains $(1, 2)$, $(5, 7)$, or $(-1, 10)$.
- An internal node with one child labelled (a, b) has label $(a, b + 1)$.
- An internal node with two childen labelled (x, y) and (a, b) has label $(\frac{x+a}{2}, \frac{y+b}{2})$.

Use (strong) induction to prove that the point in the root node of any Interp tree is above the line $x = y$

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): An Interp tree of height $h = 0$ consists of a single node containing $(1, 2)$, $(5, 7)$, or $(-1, 10)$. All three of these points are above the line $x = y$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that that the point in the root node of any Interp tree is above the line $x = y$ for trees of height $h = 0, 1, \ldots, k - 1$. $(k \geq 1)$.

Inductive Step: Let T be an Interp tree of height k There are two cases:

Case 1: the root of T has one child subtree, with label (a, b). The root of T then contains $(a, b + 1)$. By the inductive hypothesis, (a, b) is above the line $x = y$. So $b \geq a$. But then $b + 1 \geq a$. So $(a, b + 1)$ is also above the line $x = y$.

Case 2: the root of T has two child subtrees, with labels (x, y) and (a, b). The root of T then has label $(\frac{x+a}{2}, \frac{y+b}{2})$. By the inductive hypothesis (x, y) and (a, b) are above the line $x = y$. So $y \geq x$ and $b \geq a$. So $y + b \geq x + a$ and therefore $\frac{y+b}{2} \geq \frac{x+a}{2}$. So $(\frac{x+a}{2}, \frac{y+b}{2})$ is above the line $x = y$.

In both cases, the root node contains a point above the line $x = y$, which is what we needed to show.

CS 173, Spring 2015
Examlet 9, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(18 points) "Frumpy" trees are a type of tree whose nodes are labelled with integer values. Our recursive definition of frumpy trees states that a frumpy tree must be one of the following:

- A single node labelled 5 or 7.
- A tree with root labelled 0, with three frumpy trees as its children.
- A tree with root labelled 1, with two frumpy trees as its children.

The "total value" of a frumpy tree is the sum of the labels on all its nodes. Use (strong) induction that the total value of any frumpy tree is odd. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): Frumpy trees of height $h = 0$ consist of a single node containing eithe r5 or 7. So the total value of the tree is 5 or 7, both of which are odd.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that that the total value of any frumpy tree is odd, for trees of height $h = 0, 1, \ldots, k-1$. ($k \geq 1$).

Inductive Step: Let T be a frumpy tree of height k. There are two cases:

Case 1: The root of T contains 0, and it has three child subtrees. By the inductive hypothesis, the total value of each child subtree is odd. Since the sum of three odd numbers is odd, the total value of T is also odd.

Case 2: The root of T contains 1 and it has two child subtrees. By the inductive hypothesis, the total value of each child subtree is odd. The total value of T is the sum of the total values of these child subtrees, plus 1. Again, we have the sum of three odd numbers, so the total value of T must be odd.

In both cases, the total value of T is odd, which is what we needed to show.

CS 173, Spring 2015 Examlet 9, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(18 points) Here is a grammar G, with start symbol S and terminal symbols a and p.

$$S \rightarrow S\,S \;|\; p\,S\,p \;|\; p\,p \,|\; a\,a$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar G has an even number of nodes with label p. Use $P(T)$ as shorthand for the number of p's in a tree T.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest trees matching grammar G have height $h = 1$. There are two such trees, which look like

```
    S                S
  /   \            /   \
 p     p          a     a
```

Both of these contain an even number of nodes with label p.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that all trees T matching grammar G with heights $h = 1, 2, \ldots, k-1$ have $P(T)$ even, for some integer $k \geq 2$.

Inductive Step: Let T be a tree of height k matching grammar G, where $k \geq 2$. There are two cases:

Case 1: T consists of a root with label S plus two child subtrees T_1 and T_2. By the inductive hypothesis $P(T_1)$ and $P(T_2)$ are both even. But $P(T) = P(T_1) + P(T_2)$. So $P(T)$ is also even.

Case 2: T consists of a root with label S plus three children. The left and right children are single nodes containing label p. The center child is a subtree T_1. By the inductive hypothesis, $P(T_1)$ is even. $P(T) = P(T_1) + 2$. So $P(T)$ is also even.

In both cases $P(T)$ is even, which is what we needed to show.