CS 173, Spring 2015
Examlet 10, Part A

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Discussion: Monday 9 10 11 12 1 2 3 4 5

(15 points) Use (strong) induction to prove the following claim:

Claim: $n^2 < 2^n$ for any integer $n \geq 5$.

Hint: first prove that $2n + 1 \leq n^2$ for any integer $n \geq 5$. (This doesn't require induction.)

Base Case(s): At $n = 5$, $n^2 = 25 < 32 = 2^5$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $n^2 < 2^n$ for $n = 5, 6, \ldots, k$.

Inductive Step:

Since $k \geq 5$, $k \geq 1$. So $2k + 1 \leq 3k \leq 5k \leq k^2$. That is $2k + 1 \leq k^2$.

Using the above equation, we can compute $(k + 1)^2 = k^2 + (2k + 1) \leq k^2 + k^2 = 2k^2$

By the induction hypothesis $k^2 < 2^k$. Combining this with the above equation, we get $(k + 1)^2 \leq 2k^2 < 2 \cdot 2^k = 2^{k+1}$.

So $(k + 1)^2 < 2^{k+1}$ which is what we needed to prove.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{k=n+1}^{2n} \frac{1}{k} \;\geq\; \frac{7}{12}$, for any integer $n \geq 2$.

Hint: recall that if $x \leq y$, then $\frac{1}{y} \leq \frac{1}{x}$

Base Case(s): At $n = 2$, $\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{k=n+1}^{2n} \frac{1}{k} \;\geq\; \frac{7}{12}$, for $n = 2, 3, \ldots, p$.

Inductive Step: Substituing $n = p+1$ into the summation and then using the inductive hypothesis, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} \;=\; (\sum_{k=p+1}^{2p} \frac{1}{k}) + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1})$$

Now, notice that $\frac{1}{2p+1} \geq \frac{1}{2}\frac{1}{p+1}$ and $\frac{1}{2p+2} = \frac{1}{2}\frac{1}{p+1}$. So $\frac{1}{2p+1} + \frac{1}{2p+2} \geq \frac{1}{p+1}$. Therefore $\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1} \geq 0$.

Combining the results of the previous two paragraphs, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12}$$

This is what we needed to show.

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(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{N}$ be defined by

$f(1) = 0$

$f(n) = 1 + f(\lfloor n/2 \rfloor)$, for $n \geq 2$,

Use (strong) induction on n to prove that $f(n) \leq \log_2 n$ for any positive integer n. You cannot assume that n is a power of 2. However, you can assume that the log function is increasing (if $x \leq y$ then $\log x \leq \log y$) and that $\lfloor x \rfloor \leq x$.

Base Case(s):

$f(1) = 0$ and $\log_2 1 = 0$ So $f(1) \leq \log_2 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \leq log_2 n$ for $n = 1, \ldots, k-1$.

Inductive Step:

We can assume that $k \geq 2$ (since we did $n = 1$ for the base case). So $\lfloor k/2 \rfloor$ must be at least 1 and less than k. Therefore, by the inductive hypothesis, $f(\lfloor k/2 \rfloor) \leq \log_2(\lfloor k/2 \rfloor)$.

We know that $f(k) = 1 + f(\lfloor k/2 \rfloor)$, by the definition of f. Substituting the result of the previous paragraph, we get that $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$.

$\lfloor k/2 \rfloor \leq k/2$. So $\log_2(\lfloor k/2 \rfloor) \leq \log_2(k/2) = (\log_2 k) + (\log_2 1/2) = (\log_2 k) - 1$.

Since $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$ and $\log_2(\lfloor k/2 \rfloor) \leq (\log_2 k) - 1$, $f(k) \leq 1 + (\log_2 k) - 1 = (\log_2 k)$. This is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For all integers $n \geq 2$, $(2n)! > 2^n n!$

Base Case(s): At $n = 2$, $(2n)! = 4! = 24$. $2^n n! = 4 \cdot 2 = 8$. So $(2n)! > 2^n n!$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(2n)! > 2^n n!$ for all $n = 2, 3, \ldots, k$ for some integer $k \geq 2$.

Inductive Step: Notice that $2k + 1 \geq 1$ because k is positive. And $(2k)! > 2^k k!$ by the induction hypothesis.

So then

$(2(k+1))! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k)! > (2k+2)(2^k k!) = (k+1)2^{k+1}k! = 2^{k+1}(k+1)!$.

So $(2(k+1))! > 2^{k+1}(k+1)!$ which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number $x > -1$, $(1+x)^n \geq 1 + nx$.

Base Case(s): At $n = 0$, $(1+x)^n = (1+x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. So $(1+x)^n \geq 1 + nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1+x)^n \geq 1 + nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1+x)^k \geq 1 + kx$. Notice that $(1+x)$ is positive since $x > -1$. So $(1+x)^{k+1} \geq (1+x)(1+kx)$.

But $(1+x)(1+kx) = 1 + x + kx + kx^2 = 1 + (1+k)x + kx^2$.

And $1 + (1+k)x + kx^2 \geq 1 + (1+k)x$ because kx^2 is non-negative.

So $(1+x)^{k+1} \geq (1+x)(1+kx) \geq 1 + (1+k)x$, and therefore $(1+x)^{k+1} \geq 1 + (1+k)x$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \leq 2\sqrt{n}$

Hint: notice that $(\sqrt{n} - \sqrt{n+1}\ )^2 \geq 0$. What does this imply about $2\sqrt{n}\sqrt{n+1}$?

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1 \leq 2 = 2 \cdot n$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \leq 2\sqrt{n}$ for $n = 1, 2, \ldots, k$.

Inductive Step:

First, notice that $(\sqrt{k} - \sqrt{k+1}\ )^2 \geq 0$. Multiplying this out gives us $k - 2\sqrt{k}\sqrt{k+1} + (k+1) \geq 0$. So $2k + 1 \geq 2\sqrt{k}\sqrt{k+1}$.

Using this inequality plus the inductive hypothesis, we can compute

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} &= \left(\sum_{p=1}^{k} \frac{1}{\sqrt{p}}\right) + \frac{1}{\sqrt{k+1}} \\
&\leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{2\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \\
&\leq \frac{(2k+1) + 1}{\sqrt{k+1}} = \frac{2k+2}{\sqrt{k+1}} = \frac{2(k+1)}{\sqrt{k+1}} = 2\sqrt{k+1}
\end{aligned}$$