CS 173, Fall 2016
Examlet 2, Part A

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Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all real numbers p and q ($p \neq -1$), if $\frac{2}{p+1}$ and $p+q$ are rational, then q is rational.

Solution: Let p and q be real numbers, where $p \neq -1$. Suppose that $\frac{2}{p+1}$ and $p+q$ are rational.

By the definition of rational, this means that $\frac{2}{p+1} = \frac{m}{n}$ and $p+q = \frac{a}{b}$, where m, n, a, and b are integers with n and b non-zero.

Since $\frac{2}{p+1} = \frac{m}{n}$, $2n = m(p+1)$, so $p+1 = \frac{2n}{m}$. This means that $p = \frac{2n}{m} - 1 = \frac{2n-m}{m}$.

Since $p+q = \frac{a}{b}$, $q = \frac{a}{b} - p = \frac{a}{b} - \frac{2n-m}{m} = \frac{am-b(2n-m)}{bm}$.

Since a, b, n, and m are integers, $am - b(2n-m)$ and bm are both integers. Moreover, bm must be non-zero because n and b are both non-zero. So q is the ratio of two integers, with the denominator non-zero. Therefore q is rational.

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(15 points) For any two real numbers x and y, the harmonic mean is $H(x, y) = \frac{2xy}{x+y}$. Using this definition and your best mathematical style, prove the following claim:

For any real numbers x and y, if $0 < x < y$, then $H(x, y) < y$.

Solution: Let x and y be real numbers. Suppose that $0 < x < y$.

Since $x < y$ and y is positive, $xy < y^2$.

Adding xy to both sides gives us $2xy < xy + y^2 = y(x + y)$.

So $2xy < y(x + y)$. Since x and y are both positive, $x + y$ is positive. So, we can divide both sides by $(x + y)$ to get $\frac{2xy}{x+y} < y$.

So, $H(x, y) < y$, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of
congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

For all integers a, b, c, p and k (c positive), if $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$, then $k \mid b$.

Solution:

Let a, b, c, p and k be integers, with c positive. Suppose that $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$.

By the definition of congruence mod k, $ap \equiv b \pmod{c}$ implies that $ap = b + nc$ for some integer n. By the definition of divides, $k \mid a$ and $k \mid c$ imply that $a = ks$ and $c = kt$ for some integers s and t.

Since $ap = b + nc$, $b = ap - nc$. So then we have

$b = ap - nc = ksp - nkt = k(sp - nt)$

$sp - nt$ is an integer since s, p, n, and t are integers. So this implies that $k \mid b$.

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(15 points) For any two real numbers x and y, the Harmonic mean is $H(x, y) = \frac{2xy}{x+y}$. Use this definition and your best mathematical style to prove the following claim:

For all positive real numbers x and y and p, if $x \geq y$, then $H(x, p) \geq H(y, p)$.

Solution: Suppose that x, y, and p are positive reals. Suppose that $x \geq y$. Multiplying by $2p^2$, which is positive, we get $2xp^2 \geq 2yp^2$.

So $2xp^2 + 2pxy \geq 2yp^2 + 2pxy$ and thus $(2xp)(y + p) \geq (2yp)(x + p)$.

Dividing each side by the positive numbers $x + p$ and $y + p$, we get $\frac{2xp}{x+p} \geq \frac{2yp}{y+p}$. By the definition of H, this means $H(x, p) \geq H(y, p)$, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of
congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

> For all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $x^2 + xy \equiv p^2 + pq \pmod{m}$.

Solution: Let x, y, p, q and m be integers, with $m > 0$. Suppose that $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$.

By the definition of congruence mod k, this means that $x = p + am$ and $y = q + bm$, for some integers a and b. Then we can calculate

$$\begin{aligned} x^2 + xy &= (p+am)^2 + (p+am)(q+bm) \\ &= (p+am)(p+am+q+bm) \\ &= (p+am)(p+q) + (p+am)(am+bm) \\ &= (p+am)(p+q) + m(p+am)(a+b) \end{aligned}$$

Let $t = (p + am)(a + b)$. Then we have

$$\begin{aligned} x^2 + xy &= (p+am)(p+q) + mt \\ &= p(p+q) + am(p+q) + mt = p^2 + pq + m(ap + aq + t) \end{aligned}$$

$(ap + aq + t)$ is an integer because a, b, m, p, q are all integers. So $x^2 + xy \equiv p^2 + pq \pmod{m}$.

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(15 points) The following claim is slightly buggy. Explain briefly what values cause it to fail, add one extra (very simple) condition to the hypothesis. Then prove your revised claim, working directly from the definition of "divides" and using your best mathematical style.

For any integers a, b, and c, if $a^2b \mid cb$, then $a \mid c$.

Solution: This claim fails when $b = 0$. $a^2b \mid cb$ becomes $0 \mid 0$ which is true but not helpful in proving that $a \mid c$.

So let a, b, and c such that b is non-zero. Suppose that $a^2b \mid cb$.

By the definition of divides, $cb = ka^2b$ for some integer k. Since b is non-zero, we can divide both sides by b to get $c = ka^2$.

Let $p = ak$. Then $c = pa$. p must be an integer, since a and k are intgers. So, by the definition of divides, we have $a \mid c$. This is what we needed to prove.