CS 173, Fall 2016
Examlet 3, Part A

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Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

$A = \{\alpha(2, -4) + (1 - \alpha)(-2, 5)) \ | \ \alpha \in \mathbb{R}\}$

$B = \{(a, b) \in \mathbb{R}^2 \ | \ b \leq -1\}$

$C = \{(p, q) \in \mathbb{R}^2 \ | \ p \geq 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Let (x, y) be a 2D point and suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$.

Since $(x, y) \in A$, $(x, y) = \alpha(2, -4) + (1 - \alpha)(-2, 5))$ where α is a real number. So $x = 2\alpha - 2(1 - \alpha) = 4\alpha - 2$ And $y = -4\alpha + 5(1 - \alpha) = 5 - 9\alpha$

Since $(x, y) \in B$, $y \leq -1$. So we have $y = 5 - 9\alpha \leq -1$. So $6 \leq 9\alpha$. So $\alpha \geq \frac{2}{3}$.

So then $x = 4\alpha - 2 \geq 4\frac{2}{3} - 2 = \frac{8}{3} - 2 = \frac{2}{3}$.

So $x \geq 0$ and therefore $(x, y) \in C$, which is what we needed to show.

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$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 4y + 4 \leq 100\}$

$B = \{(p, q) \in \mathbb{R}^2 \mid p \leq -6\}$

$C = \{(a, b) \in \mathbb{R}^2 \mid b \leq 7\}$

Prove that $A \cap B \subseteq C$.

Solution:

Let $(x, y) \in \mathbb{R}^2$. Suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$. So $x^2 + y^2 + 4y + 4 \leq 100$ and $x \leq -6$.

Since $x \leq -6$, $x^2 \geq 36$.

Then $y^2 + 4y + 4 \leq 100 - x^2 \leq 100 - 36 = 64$.

Notice that $y^2 + 4y + 4 = (y + 2)^2$. So we have $(y + 2)^2 \leq 64$. This means $y + 2 \leq 8$. So $y \leq 6 \leq 7$.

Since $y \leq 7$, $(x, y) \in C$, which is what we needed to prove.

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$A = \{(a, b) \in \mathbb{R}^2 \ : \ a = 3 - b^2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

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For any integers s and t define $L(s,t)$ as follows:

$$L(s,t) = \{sx + ty \mid x, y \in \mathbb{Z}\}$$

Thus, $L(s,t)$ consists of all integers that can be expressed as the sum of multiples of s and t. Prove the following claim using your best mathematical style and the following definition of congruence mod k: $p \equiv q \pmod{k}$ if and only if $p = q + kn$ for some integer n.

Claim: For any integers a, b, r, where r is positive, if $a \equiv b \pmod{r}$, then $L(a,b) \subseteq L(r,b)$.

Solution: Let a, b and r be integers, where r is positive. And suppose that $a \equiv b \pmod{r}$. Then $a = b + rn$ for some integer n.

Let q be an element of $L(a,b)$. Then $q = ax + by$, where x and y are integers.

Substituting $a = b + rn$ into $q = ax + by$, we get $q = x(b + rn) + by$. So $q = (xn)r + (x + y)b$.

xn and $x + y$ are integers, because x, y, and n are integers. So $q = (xn)r + (x + y)b$ implies that $q \in L(r,b)$.

Since q was an arbitrarily chosen element of $L(a,b)$, we've shown that $L(a,b) \subseteq L(r,b)$.

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$A = \{(a, b) \in \mathbb{R}^2 \ : \ b = a^2 - 2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ \lfloor x \rfloor = 4\}$

$C = \{(p, q) \in \mathbb{R}^2 \ : \ 2p \leq q\}$

Prove that $A \cap B \subseteq$ C.

Solution: Let $(p, q) \in \mathbb{R}^2$ and suppose $(p, q) \in A \cap B$. Then $(p, q) \in A$ and $(p, q) \in B$. By the definitions of A and B, this means that $q = p^2 - 2$ and $\lfloor p \rfloor = 4$.

Since $\lfloor p \rfloor = 4$, we know that $4 \leq p < 5$.

Since $p < 5$, $2p < 10$.

Since $p \geq 4$, $q = p^2 - 2 \geq 16 - 2 = 14$.

Therefore $2p < 10 < 14 \leq q$. Since $2p \leq q$, $(p, q) \in C$, which is what we needed to show.

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$A = \{\lambda(0,3) + (1-\lambda)(2,4) \ \mid \ \lambda \in [0,1]\}$

$B = \{(x,y) \in \mathbb{R}^2 \ \mid \ x \leq y\}$

Prove that $A \subseteq B$.

Solution:

Let $(x,y) \in A$. Then $(x,y) = \lambda(0,3) + (1-\lambda)(2,4)$ for some $\lambda \in [0.1]$. So $x = 2 - 2\lambda$ and $y = 3\lambda + 4(1-\lambda) = 4 - \lambda$. So $y = x + 2 + \lambda$

Since $\lambda \in [0.1]$, $\lambda \geq 0$.

So $y = x + 2 + \lambda \geq x$.

Since $x \leq y$, $(x,y) \in B$, which is what we needed to show.