CS 173, Fall 2016
Examlet 4, Part A

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Suppose that n is some positive integer. Let's define the relation R_n on the integers such that aR_nb if and only if $a \equiv b+1 \pmod{n}$. Prove the following claim

Claim: For any integers x, y, and z, if xR_ny and yR_nz and xR_nz, then $n = 1$.

You must work directly from the definition of congruence mod k, using the following version of the definition: $x \equiv y \pmod{k}$ iff $x - y = mk$ for some integer m. You may use the following fact about divisibility: for any non-zero integers p and q, if $p \mid q$, then $|p| \leq |q|$.

Solution: Let n be a positive integer and suppose that R is as defined above. Also x, y, and z be integers and suppose that xR_ny and yR_nz and xR_nz.

By the definition of R, this means that $x \equiv y+1 \pmod{n}$, $y \equiv z+1 \pmod{n}$, and $x \equiv z+1 \pmod{n}$.

Then $x - (y+1) = kn$ $y - (z+1) = jn$, and $x - (z+1) = pn$, for some integers k, j, and p.

So then $x = y + 1 + kn$, $y = z + 1 + jn$ and $x = z + 1 + pn$. So $x = z + 2 + kn + jn$. So $z + 1 + pn = z + 2 + kn + jn$. So $pn = 1 + kn + jn$. So $(p - k - j)n = 1$.

We know that $p - k - j$ is an integer, so $(p - k - j)n = 1$ implies that $n|1$. Therefore $|n| \leq 1$. But n is known to be a positive integer. So n must equal 1.

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Let's define a relation T between pairs of natural numbers as follows:

$(a, b)T(p, q)$ if and only if $a \mid p$ and $b = q$.

Working directly from this definition and the definition of divides, prove that T is antisymmetric.

Solution: Let (a, b) and (p, q) be pairs of natural numbers. suppose that $(a, b)T(p, q)$ and $(p, q)T(a, b)$.

By the definition of T, $(a, b)T(p, q)$ implies that $a \mid p$ and $b = q$. Similarly, $(p, q)T(a, b)$ implies that $p \mid a$ and $b = q$.

Since $a \mid p$, we have $p = ak$ for some integer k. Since $p \mid a$, we have $a = pj$ for some integer j. Notice that k and j cannot be negative, because p and a are both non-negative.

Combining these two equations, we get $p = kjp$, so $kj = 1$. Since k and j are non-negative integers, we must have $k = j = 1$. So $p = ak$ implies that $a = p$.

Since $a = p$ and $b = q$, $(a, b) = (p, q)$, which is what we needed to prove.

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Let T be the relation defined on $\mathbb{N}^2$ by

$$(x, y)T(p, q) \text{ if and only if } x < p \text{ or } (x = p \text{ and } y \leq q)$$

Prove that T is transitive.

Solution:

Let (x, y), (p, q) and (m, n) be pairs of natural numbers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, $(x, y)T(p, q)$ means that $x < p$ or $(x = p$ and $y \leq q)$. Similarly $(p, q)T(m, n)$ implies that $p < m$ or $(p = m$ and $q \leq n)$.

There are four cases:

Case 1: $x < p$ and $p < m$. Then $x < m$.

Case 2: $x < p$ and $p = m$. Then $x < m$.

Case 3: $x = p$ and $p < m$. Then $x < m$.

Case 4: $x = p$ and $p = m$. In this case, we must also have $y \leq q$ and $q \leq n$. So $x = m$ and $y \leq n$.

In all four cases, $(x, y)T(m, n)$, which is what we needed to show.

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Recall how to multiply a real number α by a 2D point $(x, y) \in \mathbb{R}^2$: $\alpha(x, y) = (\alpha x, \alpha y)$.

Let $A = \mathbb{R}^+ \times \mathbb{R}^+$, i.e. pairs of positive real numbers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists a real number $\alpha \geq 1$ such that $(x, y) = \alpha(p, q)$.

Prove that $\gg$ is antisymmetric.

Solution: Let (x, y) and (p, q) be elements of A. Suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (x, y)$.

By the definition of $\gg$, there are real numbers $\alpha \geq 1$ and $\beta \geq 1$ such that $(x, y) = \alpha(p, q)$ and $(p, q) = \beta(a, b)$.

Substituting the second equation into the first, we get $(x, y) = \alpha\beta(x, y)$. This means that $\alpha\beta = 1$. Since $\alpha \geq 1$ and $\beta \geq 1$, this implies that $\alpha = \beta = 1$. So therefore $(x, y) = (p, q)$, which is what we needed to show.

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Let's define a relation T between natural numbers follows:

aTb if and only if $a = b + 2k$ where $k \in \mathbb{N}$

Working directly from this definition, prove that T is antisymmetric.

Solution: Let a and b be natural numbers and suppose that aTb and bTa.

By the definition of T, this means that $a = b+2k$ and $b = a+2j$, where k and j are natural numbers.

Substituting one equation into the other, we get $a = (a + 2j) + 2k = a + 2(j + k)$. So $2(j + k) = 0$. So $j + k = 0$.

Notice that j and k are both non-negative. So $j + k = 0$ implies that $j = k = 0$.

So $a = b$, which is what we needed to show.

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $(xy)(p + q) = (pq)(x + y)$

Prove that T is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $(xy)(p + q) = (pq)(x + y)$ and $(pq)(m + n) = (mn)(p + q)$

Since $m+n$ is positive, we can divide both sides by it, to get $(pq) = (mn)(p+q)/(m+n)$. Substituting this into the first equation, we get

$$(xy)(p + q) = (mn)(p + q)/(m + n) \times (x + y)$$

Multiplying both sides by $(m + n)$, we get

$$(xy)(p + q)(m + n) = (mn)(p + q)(x + y)$$

Since $(p + q)$ is positive, we can cancel it from both sides to get

$$(xy)(m + n) = (mn)(x + y)$$

By the definition of T, this means that $(a, b)T(m, n)$, which is what we needed to show.