CS 173, Fall 2016
Examlet 5, Part A

NETID:

FIRST: | **LAST:**

Discussion: **Thursday 2 3 4 5 Friday 9 10 11 12 1 2**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (2f(x) + f(y), f(x) - f(y))$. Prove that g is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $g(x, y) = g(p, q)$.

   By the definition of h, this means that $(2f(x) + f(y), f(x) - f(y)) = (2f(p) + f(q), f(p) - f(q))$. So $2f(x) + f(y) = 2f(p) + f(q)$ and $f(x) - f(y) = f(p) - f(q)$.

   Adding these two equations, we get $3f(x) = 3f(p)$. So $f(x) = f(p)$. Since f is one-to-one, this means that $x = p$.

   Subtracting twice the second equation from the first, we get $-3f(y) = -3f(q)$. So $f(y) = f(q)$. Since f is one-to-one, this means that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) $A = \{1, 3, 5, 7, 9, \ldots\}$, i.e. the positive odd numbers.

   $B = \{-1, -2, -3, -4, -5 \ldots\}$, i.e. negative numbers

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = -\frac{n+1}{2}$

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1. (10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

   By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

   Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

   Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

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1. (10 points) Suppose that $f : [0, \frac{1}{2}] \to [1, \frac{5}{2}]$ is defined by $f(x) = \frac{x^2+1}{1-2x^2}$ Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution:

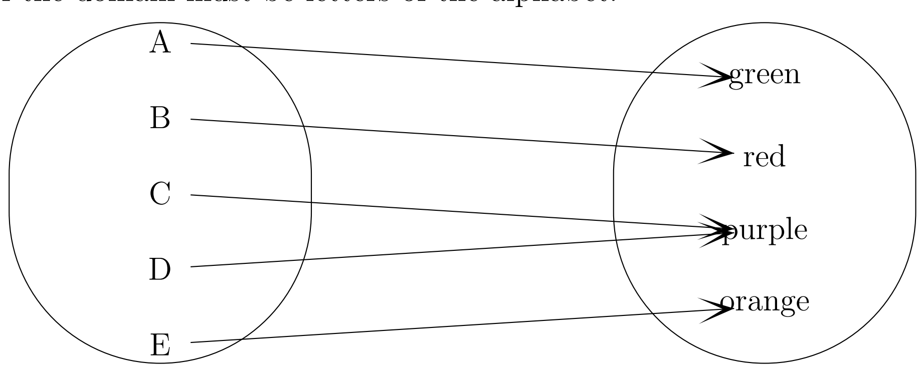
   Let x and y be any numbers in $[0, \frac{1}{2}]$ and suppose $f(x) = f(y)$, that is

$$
\begin{aligned}
& \frac{x^2+1}{1-2x^2} = \frac{y^2+1}{1-2y^2} \\
\Rightarrow\ & (x^2+1)(1-2y^2) = (y^2+1)(1-2x^2) \\
\Rightarrow\ & x^2+1-2x^2y^2-2y^2 = y^2+1-2x^2y^2-2x^2 \\
\Rightarrow\ & 3x^2 = 3y^2 \\
\Rightarrow\ & x = y
\end{aligned}
$$

   (The last step works because x and y are both positive.)

   Therefore f is one-to-one.

2. (5 points) Complete this picture to make an example of a function that is onto but not one-to-one, by adding elements to the domain and arrows showing how input values map to output values. The elements of the domain must be letters of the alphabet.



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1. (10 points) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $h : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $h(x, y) = (g(x) + g(y), g(x) - g(y))$. Prove that h is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $h(x, y) = h(p, q)$.

   By the definition of h, this means that $(g(x) + g(y), g(x) - g(y)) = (g(p) + g(q), g(p) - g(q))$. So $g(x) + g(y) = g(p) + g(q)$ and $g(x) - g(y) = g(p) - g(q)$.

   Adding these equations together, we get $2g(x) = 2g(p)$. So $g(x) = g(p)$. Since g is one-to-one, this implies that $x = p$.

   Similarly, if we subtract the two equations, we get $2g(y) = 2g(q)$. So $g(y) = g(q)$. And since g is one-to-one, $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is onto but not one-to-one. Be specific.

   Solution: Let $f(n) = \lfloor n/2 \rfloor$ Then f is onto. But f isn't one-to-one because (for example) both 0 and 1 are mapped onto 0.

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1. (10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = xy + yx^2 - x^2$. Prove that f is onto.

   Solution:

   Notice that $f(x, y) = xy + (y - 1)x^2$.

   Let p be an integer. We need to find a pre-image for p.

   Consider $m = (p, 1)$.

   m is an element of $\mathbb{Z}^2$. We can compute

   $$f(m) = p \cdot 1 + (1 - 1)p^2 = p + 0 \cdot p^2 = p$$

   So m is a pre-image of p.

   Since we can find a pre-image for an arbitrarily chosen integer, f is onto.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that f is onto.

   Solution: Let m be an element of A. We need to find a pre-image for m.

   Consider $n = g(m)$. n is an element of B. Furthermore, since $f(g(x)) = x$ for every $x \in A$, we have $f(n) = f(g(m)) = m$.

   So n is a pre-image of m.

   Since we can find a pre-image for an arbitrarily chosen element of A, f is onto.

2. (5 points) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is one-to-one, then f is one-to-one. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

   Solution: