CS 173, Fall 2016
Examlet 7, Part A

NETID:

FIRST:	**LAST:**

Discussion: **Thursday 2 3 4 5 Friday 9 10 11 12 1 2**

Use (strong) induction to prove the following claim:

For any natural number n, $2n^3 + 3n^2 + n$ is divisible by 6.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $2n^3 + 3n^2 + n = 0$ which is divisible by 6.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $2n^3 + 3n^2 + n$ is divisible by 6 for $n = 0, 1, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $2k^3 + 3k^2 + k$ is divisible by 6.

$$\begin{aligned}
2(k+1)^3 + 3(k+1)^2 + (k+1) &= 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + (k+1) \\
&= 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 + k + 1 \\
&= 2k^3 + 6k^2 + 6k + 3k^2 + 6k + k + 6 \\
&= 2k^3 + 3k^2 + k + 6k + 6 + 6k^2 + 6k \\
&= (2k^3 + 3k^2 + k) + 6(k^2 + 2k + 1)
\end{aligned}$$

$2k^3 + 3k^2 + k$ is divisible by 6 by the inductive hypothesis. $6(k^2 + 2k + 1)$ is divisible by 6 because $k^2 + 2k + 1$ is an integer (since k is an integer).

So $2(k+1)^3 + 3(k+1)^2 + (k+1)$ is the sum of two multiples of 6 and therefore divisible by 6.

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Use (strong) induction to prove the following claim:

Claim: $\displaystyle\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$ for all integers $n \geq 1$.

Solution:

Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{1}{1 \cdot 3} = \frac{1}{3}$. Also $\frac{n}{2n+1} = \frac{1}{3}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$ for $n = 1, \ldots, k$ for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)}$.

By removing the top term of the summation and then applying the inductive hypothesis, we get

$\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{1}{(2(k+1)-1)(2(k+1)+1)} + \sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} = \frac{1}{(2k+1)(2k+3)} + \frac{k}{2k+1}$

Adding the two fractions together:

$\frac{1}{(2k+1)(2k+3)} + \frac{k}{2k+1} = \frac{1}{(2k+1)(2k+3)} + \frac{k(2k+3)}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

So $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{k+1}{2k+3}$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $3^{2n+1} + 1$ is divisible by 4, for all natural numbers n

Solution: Proof by induction on n.

Base case(s): $n = 0$. At $n = 0$, $3^{2n+1} + 1 = 3^1 + 1 = 4$ which is clearly divisible by 4.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

$3^{2n+1} + 1$ is divisible by 4 for $n = 0, 1, \ldots, k$ for some integer $k \geq 0$.

Rest of the inductive step: In particular, by the inductive hypothesis, we know that $3^{2k+1} + 1$ is divisible by 4. So $3^{2k+1} + 1 = 4p$ for some integer p. So $3^{2k+1} = 4p - 1$.

Consider $3^{2(k+1)+1} + 1$. $3^{2(k+1)+1} + 1 = 3^{2k+3} + 1 = 9 \cdot 3^{2k+1} + 1$

Substituting in $3^{2k+1} = 4p - 1$, we get $9 \cdot 3^{2k+1} + 1 = 9 \cdot (4p - 1) + 1 = 36p - 9 + 1 = 36p - 8 = 4(9p - 2)$.

So $3^{2(k+1)+1} + 1 = 4(9p - 2)$. $9p - 2$ is an integer since p is an integer. So this means that $3^{2(k+1)+1} + 1$ is divisible by 4, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$ for all integers $n \geq 2$.

Solution:

Proof by induction on n.

Base case(s): $n = 2$. At $n = 2$, $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{1}{2}$. Also $\frac{n-1}{n} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$. for $n = 2, \ldots, k$ for some integer $k \geq 2$.

Rest of the inductive step:

Consider $\sum_{j=2}^{k+1} \frac{1}{j(j-1)}$.

By removing the top term of the summation and then applying the inductive hypothesis, we get $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \sum_{j=2}^{k} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \frac{k-1}{k}$.

Adding the two fractions together:

$\frac{1}{(k+1)k} + \frac{k-1}{k} = \frac{1}{(k+1)k} + \frac{(k+1)(k-1)}{(k+1)k} = \frac{1}{(k+1)k} + \frac{k^2-1}{(k+1)k} = \frac{k^2}{(k+1)k} = \frac{k}{(k+1)}$

So $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{k}{(k+1)}$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $7^n - 2^n$ is divisble by 5, for all natural numbers n.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $7^n - 2^n = 1 - 1 = 0$, which is divisible by 5.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

$7^n - 2^n$ is divisble by 5, $n = 0, 1, \ldots, k$ for some integer $k \geq 0$.

Rest of the inductive step:

Consider $7^{k+1} - 2^{k+1}$. We need to prove that it is divisible by 5.

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k = 2 \cdot 7^k + 5 \cdot 7^k - 2 \cdot 2^k \\ &= 2(7^k - 2^k) + 5 \cdot 7^k \end{aligned}$$

By the inductive hypothesis $7^k - 2^k$ is divisible by 5. So $2(7^k - 2^k)$ is divisble by 5. $5 \cdot 7^k$ is obviously divisble by 5.

$7^{k+1} - 2^{k+1}$ is the sum of two terms that are divisble by 5, so $7^{k+1} - 2^{k+1}$ is divisble by 5. This is what we needed to prove.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} j(j+1) = 1(1+1) = 2$ Also, $\frac{n(n+1)(n+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$. So the two sides of the equation are equal at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for $n = 1, \ldots, k$, for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} j(j+1)$. By removing the top term of the summation and applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} j(j+1) = (k+1)(k+2) + \sum_{j=1}^{k} j(j+1) = (k+1)(k+2) + \frac{k(k+1)(k+2)}{3}$$

Simplifying the algebra:

$(k+1)(k+2) + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)}{3} + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2) \ + \ k(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$

So $\sum_{j=1}^{k+1} j(j+1) = \frac{(k+1)(k+2)(k+3)}{3}$, which is what we needed to show.