CS 173, Fall 2016
Examlet 8, Part A

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(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 2$

$f(1) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$, for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$
\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 2(3 \cdot 2^{k-2} + (-1)^{k-1}) && \text{by inductive hypothesis} \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 3 \cdot 2^{k-1} + 2(-1)^{k-1} \\
&= 6 \cdot 2^{k-1} + (-1)^k - 2(-1)^k \\
&= 3 \cdot 2^k - (-1)^k \\
&= 3 \cdot 2^k (-1)^{k+1}
\end{aligned}
$$

So $f(k) = 3 \cdot 2^k (-1)^{k+1}$, which is what we needed to show.

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(20 points) Use (strong) induction to prove that, for any integer $n \geq 8$, there are non-negative integers p and q such that $n = 3p + 5q$.

Solution: Proof by induction on n.

Base case(s): At $n = 8$, we can chose $p = 1$ and $q = 1$. At $n = 9$, we can chose $p = 3$ and $q = 0$. At $n = 10$, we can chose $p = 0$ and $q = 2$. In all three cases, $n = 3p + 5q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there are non-negative integers p and q such that $n = 3p + 5q$, for $n = 8, 9, \ldots, k-$, where $k \geq 11$.

Rest of the inductive step: Consider $n = k$.

Notice that $k \geq 11$, so $8 \leq k - 3 \leq k - 1$. So $k - 3$ is covered by the inductive hypothesiss. Therefore, there are non-negative integers r and q such that $k - 3 = 3r + 5q$.

Now, set $p = r + 1$. Then $k = (k - 3) + 3 = (3r + 5q) + 3 = 3(r + 1) + 5q = 3p + 5q$. p is non-negative since r is.

So there are non-negative integers p and q such that $k = 3p + 5q$, which is what we needed to prove.

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(20 points) Recall that the hypercube Q_2 is a 4-cycle, and that Q_n consists of two copies of Q_{n-1} plus edges connecting corresponding nodes. A *Hamiltonian cycle* is a cycle that visits each node exactly once, except obviously for when it returns to the starting node at the end. Use (strong) induction to show Q_n has Hamiltonian cycle for any natural number $n \geq 2$.

Solution: Proof by induction on n.

Base case(s): At $n = 2$, Q_2 is the same as (isomorphic to) C_4. The entire graph forms a Hamiltonian cycle.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that Q_n has a Hamiltonian cycle for $n = 2, 3, \ldots, k-1$.

Rest of the inductive step: Consider Q_k. Q_k consists of two copies of Q_{k-1}, plus the connecting edges of the form xx' where x and x' are corresponding nodes in the two copies. Each of the smaller hypercubes has a Hamiltonian cycle by the inductive hypothesis. Remove an edge ab from one of these cycles and the corresponding edge $a'b'$ from the other cycle. Next, join the two partial cycles using the connector edges aa' and bb' This new cycle is a Hamiltonian cycle for Q_k.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 0 \qquad f(2) = 12$

$f(n) = 4f(n-1) - 3f(n-2), \quad \text{for } n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot 3^n - 6$

Solution: Proof by induction on n.

Base case(s): For $n = 1$, $f(1) = 0$ and $2 \cdot 3^n - 6 = 2\dot{3} - 6 = 0$. So the claim is true.

For $n = 2$, $f(2) = 12$ and $2 \cdot 3^n - 6 = 2\dot{3}^2 - 6 = 18 - 6 = 12$. So the claim is true.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 2 \cdot 3^n - 6$ for $n = 1, 2, \ldots, k-1$ for some positive integer $k \geq 3$.

Rest of the inductive step:

$f(k) = 4 \cdot f(k-1) - 3 \cdot f(k-2)$ by the definition of f.

So $f(k) = 4 \cdot (2 \cdot 3^{k-1} - 6) - 3 \cdot (2 \cdot 3^{k-2} - 6)$ by the inductive hypothesis.

So $f(k) = 8 \cdot 3^{k-1} - 24 - 6 \cdot 3^{k-2} + 18 = 8 \cdot 3^{k-1} - 2 \cdot 3^{k-1} - 6 = 6 \cdot 3^{k-1} - 6 = 2 \cdot 3^k - 6$

So $f(k) = 2 \cdot 3^k - 6$ which is what we needed to show.

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(20 points) A Zellig graph has $2n$ nodes arranged in a circle. Half of the nodes have label 1 and the other half have label -1. As you move clockwise around the circle, you keep a running total of node labels. E.g. if you start at a 1 node and then pass through two -1 nodes, your running total is -1. Use (strong) induction to prove that there is a choice of starting node for which the running total stays ≥ 0.

Hint: remove an adjacent pair of nodes.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, there are only two nodes. If you start at the node with label 1, the running total stays ≥ 0.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there is a choice of starting node for which the running total stays ≥ 0, for Zellig graphs with $2n$ nodes, where $n = 1, \ldots, k-1$.

Rest of the inductive step: Let G be a Zellig graph with $2k$ nodes. Find a 1 node that immediately precedes a -1 (going clockwise). Remove those two nodes m and n from G to create a smaller graph H.

By the inductive hypothesis, we can find a starting node p on H such that the running total stays ≥ 0. I claim that p also works as a starting node for G. Between p and m, we see the same sequence of nodes as in H, so the total stays ≥ 0. The total increases by 1 at m and the immediately decreases by 1 at n. So it can't dip below zero in that section of the circle. Between n and returning to p, we have the same running totals as in H.

So G has a starting point for which all the running totals stay ≥ 0, which is what we needed to prove.

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(20 points) Use (strong) induction to prove that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an integer for all natural numbers n

Hint: $(a^n + b^n)(a+b) = (a^{n+1} + b^{n+1}) + ab(a^{n-1} + b^{n-1})$, for any real numbers a and b.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $(3+\sqrt{5})^n + (3-\sqrt{5})^n = 1 + 1 = 2$, which is an integer.

At $n = 1$, $(3+\sqrt{5})^n + (3-\sqrt{5})^n = (3+\sqrt{5}) + (3-\sqrt{5}) = 6$, which is an integer.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an integer for $n = 0, 1, \dots, k$.

Rest of the inductive step:

Let $a = (3+\sqrt{5})$ and $a = (3-\sqrt{5})$. Then the inductive hypothesis tells us that $a^k + b^k$ is an integer, and $a^{k-1} + b^{k-1}$ is an integer.

Notice also that $a + b = 6$ and $ab = (3+\sqrt{5})(3-\sqrt{5}) = 9 - 5 - 4$.

Now, using the hint, we can calculate

$$\begin{aligned}
(3+\sqrt{5})^{k+1} + (3-\sqrt{5})^{k+1} &= a^{k+1} + b^{k+1} \\
&= (a^k + b^k)(a+b) - ab(a^{k-1} + b^{k-1}) \\
&= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1})
\end{aligned}$$

The righthand expression $6(a^k+b^k) - 4(a^{k-1}+b^{k-1})$ must be an integer because it's made by multiplying and subtracting integers. So the lefthand expression, i.e. $(3+\sqrt{5})^{k+1} + (3-\sqrt{5})^{k+1}$ must be an integer, whcih is what we needed to show.