CS 173, Fall 2016
Examlet 8, Part B

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Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= c \\ F(n) &= F(n/2) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$F(n) = F(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i}$$

Finish finding the closed form for F. Show your work and simplify your answer.

Solution:

To find the value of k at the base case, we need to set $n/2^k = 2$. This means that $n = 2 \cdot 2^k$. So $n = 2^{k+1}$. So $k + 1 = \log n$. So $k = \log n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} T(n) &= T(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i} = T(4) + \sum_{i=0}^{\log n - 2} n\frac{1}{2^i} \\ &= c + n\sum_{i=0}^{\log n - 2} \frac{1}{2^i} = c + n(2 - \frac{1}{2^{\log n - 2}}) \\ &= c + n(2 - \frac{1}{2^{\log n} \cdot 2^{-2}}) = c + n(2 - \frac{4}{2^{\log n}}) \\ &= c + n(2 - \frac{4}{n}) = c + 2n - 4 \end{aligned}$$

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1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + d \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + d \\ &= 4(4g(n/2^2 + d) + d \\ &= 4(4(4g(n/2^3) + d) + d) + d \\ &= 4^3 g(n/2^3) + 21d \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The number of edges in the 4-dimensional hypercube Q_4	5 ☐	12 ☐	32 ☑	64 ☐

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1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n/2) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 3f(n/2) + n^2 \\ &= 3(3f(n/4) + (n/2)^2) + n^2 \\ &= 3(3(3f(n/8) + (n/4)^2) + (n/2)^2) + n^2 \\ &= 27f(n/2^3) + (3/4)n^2 + (9/16)n^2 + n^2 \\ &= 27f(n/2^3) + (2 + 5/16)n^2 \end{aligned}$$

2. (2 points) The n-dimensional hypercube Q_n has an Euler circuit.

always ☐ sometimes ☑ never ☐

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(10 points) Suppose we have a function F defined (for n a power of 3) by

$$\begin{aligned} F(1) &= 5 \\ F(n) &= 3F(n/3) + 7 \text{ for } n \geq 3 \end{aligned}$$

Your partner has already figured out that

$$F(n) = 3^k F(n/3^k) + 7 \sum_{p=0}^{k-1} 3^p$$

Finish finding the closed form for F. Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

Solution:

To find the value of k at the base case, set $n/3^k = 1$. Then $n = 3^k$, so $k = \log_3 n$. Substituting this into the above, we get

$$\begin{aligned} F(n) &= 3^{\log_3 n} \cdot 5 + 7 \sum_{p=0}^{\log_3 n - 1} 3^p \\ &= 5n + 7\frac{3^{log_3 n} - 1}{3-1} \\ &= 5n + 7\frac{n-1}{3-1} = 5n + \frac{7(n-1)}{2} \end{aligned}$$

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Discussion: **Thursday 2 3 4 5 Friday 9 10 11 12 1 2**

(10 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 2^k g(n/4^k) + n \sum_{p=0}^{k-1} \frac{1}{2^p}$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 2^{\log_4 n} \cdot c + n \sum_{p=0}^{\log_4 n - 1} \frac{1}{2^p} \\ &= 2^{\log_4 n} \cdot c + n(2 - \frac{1}{2^{\log_4 n - 1}}) \\ &= c\sqrt{n} + n(2 - \frac{2}{\sqrt{n}}) \\ &= c\sqrt{n} + 2n - 2\sqrt{n} \\ &= 2n + (c-2)\sqrt{n} \end{aligned}$$

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1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(9) &= 5 \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 27 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 3^k g(n/3^k) + kn$$

Finish finding the closed form for g. Show your work and simplify your answer.

Solution:

To find the value of k at the base case, set $n/3^k = 9$. Then $n = 3^{k+2}$, so $k + 2 = \log_3 n$, so $k + 2 = \log_3 n - 2$ Substituting this into the above, we get:

$$\begin{aligned} g(n) &= 3^{log_3 n - 2} g(9) + (log_3 n - 2)n \\ &= 3^{log_3 n} 3^{-2} 5 + n \log_3 n - 2n) \\ &= \frac{5}{9} n + n \log_3 n - 2n = n \log_3 n - \frac{13}{9} n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

$f(n) = n!$ can be defined recursively by $f(0) = 1$, and $f(n+1) = (n+1)f(n)$ for all integers ...	$n \geq 0$ ☑	$n \geq 1$ ☐	$n \geq 2$ ☐