CS 173, Fall 2016
Examlet 9, Part A

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(18 points) If T is a binary tree with root R, then Heft(T) is defined to be

- 0 if R is a leaf
- m if R has one child subtree T', with Heft(T') equal to m
- 1+m if R has two child subtrees V and W, with Heft(V) and Heft(W) both equal to m
- otherwise, the maximum Heft of R's two child subtrees.

Use (strong) induction to prove that a binary tree T with Heft(T)=p has at least 2^p leaves

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): $h = 0$. In this case, the tree has only one node, so its Heft is 0. And it has $2^0 = 1$ leaves. So the claim is true for h=0.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Any binary tree T with Heft(T)=p has at least 2^p leaves, for trees of height $h = 0, 1, \ldots, k-1$, $(k \geq 1)$.

Inductive Step: Let T be a tree of height k. There are three cases:

Case 1: T has one child subtree T_1 with Heft(T_1) = Heft(T) = q. By the inductive hypothesis T_1 has at least 2^q leaves, and T has the same number of leaves. So T also has at least 2^q leaves.

Case 2: T has two child subtrees T_L and T_R, where Heft(T_L)=Heft(T_M)=m and Heft(T)=m+1. By the inductive hypothesis T_L has at least 2^m leaves, and T_R has at least 2^m leaves. So T has at least 2^{m+1} leaves.

Case 3: T has two child subtrees T_L and T_R, where Heft(T_L)=m, Heft(T_M)=n, and Heft(T)=max(m,n). By the inductive hypothesis T_L has at least 2^m leaves, and T_R has at least 2^n leaves. So T has at least $2^{max(m,n)}$ leaves.

In all three cases, T has at least 2^q leaves, where q=Heft(T). This is what we needed to prove.

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(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Happy tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Happy tree of height h contains at least F_{h+1} nodes, where F_k is the kth Fibonacci number. (Recall: $F_0 = 0$, $F_1 = F_2 = 1$)

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): At $h = 0$, the tree contains exactly one node and therefore exactly one leaf. Also, $F_{h+1} = F_1 = 1$, so the claim holds.

At $h = 1$, the tree must consist of three nodes: a root and its two children. So it has three nodes. $F_{h+1} = F_2 = 1$. So the number of leaves is $\geq F_{h+1}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that every Happy tree of height h contains at least F_{h+1} nodes, for $h = 0, 1, \ldots, k-1$.

Inductive Step: Let T be a Happy tree of height k ($k \geq 1$). The root of T must have two child subtrees T_a and T_b, whose heights differ by at most one.

Case 1: T_a has height $k-1$ and T_b has height $k-2$. By the inductive hypothesis, T_a has at least F_k nodes and T_b has at least F_{k-1} nodes. So T must have at least $F_k + F_{k-1} = F_{k+1}$ nodes.

Case 2: T_a has height $k-2$ and T_b has height $k-1$. By the inductive hypothesis, T_a has at least F_{k-1} nodes and T_b has at least F_k nodes. So T must have at least $F_k + F_{k-1} = F_{k+1}$ nodes.

Case 3: T_a and T_b have height $k-1$. By the inductive hypothesis, T_a and T_b each have at least F_k nodes. So T must have at least $2F_k \geq F_k + F_{k-1} = F_{k+1}$ nodes.

In all cases, T must have at least F_{k+1} nodes, which is what we needed to prove.

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(18 points) A Strange tree is a 3-ary tree (i.e. nodes have between 0 and 3 children) whose nodes are labelled with natural numbers such that

- Leaf nodes have label 1.
- The label on an internal node is the sum of the labels on its children, plus one.

Use (strong) induction to prove that the root node of any Strange tree has label $\leq \frac{1}{2}(3^{h+1}-1)$ where h is the height of the tree.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): At height 0, the tree consists of a single node with label 1. $\frac{1}{2}(3^{h+1}-1) = 1$ so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the root node of any Strange tree has label $\leq \frac{1}{2}(3^{h+1}-1)$ where h is the height of the tree, for $h = 0, \ldots, k-1$, where $k \geq 1$.

Inductive Step: Let T be a Strange tree of height $k \geq 1$. There are three cases:

Case 1: The root of T has a single child with label m. Then the root of T has label $m+1$. By the inductive hypothesis. $m \leq \frac{1}{2}(3^k-1)$. So $m+1 \leq \frac{1}{2}(3^k-1)+1$.

Case 2: The root of T has two children with labels m and n Then the root of T has label $m+n+1$. By the inductive hypothesis. $m \leq \frac{1}{2}(3^k-1)$ and $n \leq \frac{1}{2}(3^k-1)$. So $m+n+1 \leq 2\frac{1}{2}(3^k-1)+1$.

Case 3: The root of T has three children with labels m, n, and p. Then the root of T has label $m+n+p+1$. By the inductive hypothesis. $m \leq \frac{1}{2}(3^k-1)$, $n \leq \frac{1}{2}(3^k-1)$, and $p \leq \frac{1}{2}(3^k-1)$. So $m+n+p+1 \leq 3\frac{1}{2}(3^k-1)+1$.

Notice that $\frac{1}{2}(3^k-1)+1 \leq 2\frac{1}{2}(3^k-1)+1 \leq 3\frac{1}{2}(3^k-1)+1$. So, in all three cases, the root label of T is $\leq 3\frac{1}{2}(3^k-1)+1$.

But $3\frac{1}{2}(3^k-1)+1 = \frac{1}{2}(3^{k+1}-3)+1 = \frac{1}{2}(3^{k+1}-1)$ So, in all three cases the root label of T is $\leq \frac{1}{2}(3^{k+1}-1)$ This is what we needed to prove.

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(18 points) Let's define a Spooky Tree to be a binary tree containing 2D points such that:

- Each leaf node contains $(1, 2)$, $(5, 7)$, or $(-1, 10)$.
- An internal node with one child labelled (a, b) has label $(a, b + 1)$.
- An internal node with two childen labelled (x, y) and (a, b) has label $(\frac{x+a}{2}, \frac{y+b}{2})$.

Use (strong) induction to prove that the point in the root node of any Spooky tree is above the line $x = y$

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): A Spooky tree of height $h = 0$ consists of a single node containing $(1, 2)$, $(5, 7)$, or $(-1, 10)$. All three of these points are above the line $x = y$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that that the point in the root node of any Spooky tree is above the line $x = y$ for trees of height $h = 0, 1, \ldots, k - 1$. $(k \geq 1)$.

Inductive Step: Let T be a Spooky tree of height k There are two cases:

Case 1: the root of T has one child subtree, with label (a, b). The root of T then contains $(a, b + 1)$. By the inductive hypothesis, (a, b) is above the line $x = y$. So $b \geq a$. But then $b + 1 \geq a$. So $(a, b + 1)$ is also above the line $x = y$.

Case 2: the root of T has two child subtrees, with labels (x, y) and (a, b). The root of T then has label $(\frac{x+a}{2}, \frac{y+b}{2})$. By the inductive hypothesis (x, y) and (a, b) are above the line $x = y$. So $y > x$ and $b > a$. So $y + b \geq x + a$ and therefore $\frac{y+b}{2} > \frac{x+a}{2}$. So $(\frac{x+a}{2}, \frac{y+b}{2})$ is above the line $x = y$.

In both cases, the root node contains a point above the line $x = y$, which is what we needed to show.

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(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Peaceful tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Peaceful tree of height h contains at least F_{h+1} nodes, where F_k is the kth Fibonacci number. (Recall: $F_0 = 0$, $F_1 = F_2 = 1$)

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): At $h = 0$, the tree contains exactly one node and therefore exactly one leaf. Also, $F_{h+1} = F_1 = 1$, so the claim holds.

At $h = 1$, the tree must consist of three nodes: a root and its two children. So it has three nodes. $F_{h+1} = F_2 = 1$. So the number of leaves is $\geq F_{h+1}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that every Peaceful tree of height h contains at least F_{h+1} nodes, for $h = 0, 1, \ldots, k-1$.

Inductive Step: Let T be a Peaceful tree of height k ($k \geq 1$). The root of T must have two child subtrees T_a and T_b, whose heights differ by at most one.

Case 1: T_a has height $k-1$ and T_b has height $k-2$. By the inductive hypothesis, T_a has at least F_k nodes and T_b has at least F_{k-1} nodes. So T must have at least $F_k + F_{k-1} = F_{k+1}$ nodes.

Case 2: T_a has height $k-2$ and T_b has height $k-1$. By the inductive hypothesis, T_a has at least F_{k-1} nodes and T_b has at least F_k nodes. So T must have at least $F_k + F_{k-1} = F_{k+1}$ nodes.

Case 3: T_a and T_b have height $k-1$. By the inductive hypothesis, T_a and T_b each have at least F_k nodes. So T must have at least $2F_k \geq F_k + F_{k-1} = F_{k+1}$ nodes.

In all cases, T must have at least F_{k+1} nodes, which is what we needed to prove.

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(18 points) Here is a grammar G, with start symbol S and terminal symbols a and p.

$$S \rightarrow S\ S \mid p\ S\ p \mid p\ p \mid a\ a$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar G has an even number of nodes with label p. Use $P(T)$ as shorthand for the number of p's in a tree T.

Solution: The induction variable is named <u>**h**</u> and it is the <u>**height**</u> of/in the tree.

Base Case(s): The shortest trees matching grammar G have height $h = 1$. There are two such trees, which look like

Both of these contain an even number of nodes with label p.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that all trees T matching grammar G with heights $h = 1, 2, \ldots, k-1$ have $P(T)$ even, for some integer $k \geq 2$.

Inductive Step: Let T be a tree of height k matching grammar G, where $k \geq 2$. There are two cases:

Case 1: T consists of a root with label S plus two child subtrees T_1 and T_2. By the inductive hypothesis $P(T_1)$ and $P(T_2)$ are both even. But $P(T) = P(T_1) + P(T_2)$. So $P(T)$ is also even.

Case 2: T consists of a root with label S plus three children. The left and right children are single nodes containing label p. The center child is a subtree T_1. By the inductive hypothesis, $P(T_1)$ is even. $P(T) = P(T_1) + 2$. So $P(T)$ is also even.

In both cases $P(T)$ is even, which is what we needed to show.