CS 173, Fall 2016
Examlet 10, Part A

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Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Use (strong) induction to prove the following claim:

Claim: $(2n)!^2 < (4n)!$ for all positive integers.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(2n)!^2 = (2!)^2 = 2^2 = 4$ And $(4n)! = 4! = 24$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(2n)!^2 < (4n)!$ for $n = 1, 2, \ldots, k$.

Inductive Step: At $n = k + 1$, we have

$(2(k+1))!^2 = (2k+2)!^2 = [(2k+2)(2k+1)(2k!)]^2 = (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2$

Also $(4(k+1))! = (4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$

Also notice that $(2k+2)(2k+2)(2k+1)(2k+1) < (4k+4)(4k+3)(4k+2)(4k+1)$ because each of the four terms on the left is smaller than the four terms on the right.

From the inductive hypothesis, we know that $(2k)!^2 < (4k)!$.

Putting this all together, we get

$$\begin{aligned}
(2(k+1))!^2 &= (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2 \\
&< (2k+2)(2k+2)(2k+1)(2k+1)(4k)! \\
&< (4k+4)(4k+3)(4k+2)(4k+1)(4k)! \\
&= (4(k+1))!
\end{aligned}$$

So $(2(k+1))!^2 < (4(k+1))!$, which is what we needed to prove.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number $x > -1$, $(1+x)^n \geq 1 + nx$.

Let x be a real number with $x > -1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1+x)^n = (1+x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. So $(1+x)^n \geq 1 + nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1+x)^n \geq 1 + nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1+x)^k \geq 1 + kx$. Notice that $(1+x)$ is positive since $x > -1$. So $(1+x)^{k+1} \geq (1+x)(1+kx)$.

But $(1+x)(1+kx) = 1 + x + kx + kx^2 = 1 + (1+k)x + kx^2$.

And $1 + (1+k)x + kx^2 \geq 1 + (1+k)x$ because kx^2 is non-negative.

So $(1+x)^{k+1} \geq (1+x)(1+kx) \geq 1 + (1+k)x$, and therefore $(1+x)^{k+1} \geq 1 + (1+k)x$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$

You may use the fact that $\sqrt{n+1} \geq \sqrt{n}$ for any natural number n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$ Also $\sqrt{n} = 1$. So $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$ for $n = 1, 2, \ldots, k$, for some integer $k \geq 1$.

Inductive Step: $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \sqrt{k}$ by the inductive hypothesis.

So

$$\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} = \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \frac{1}{\sqrt{k+1}} + \sqrt{k} = \frac{1 + \sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} \geq \frac{1 + \sqrt{k}\sqrt{k}}{\sqrt{k+1}} = \frac{1+k}{\sqrt{k+1}} = \sqrt{k+1}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \geq \sqrt{k+1}$, which is what we needed to show.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$.

Use (strong) induction to prove the following claim:

Claim: For every integer $n \geq 2$, $\displaystyle\prod_{p=1}^{n} \frac{2p-1}{2p} > \frac{1}{2\sqrt{n}}$

You may use the fact that $\sqrt{2} > 1.4$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\prod_{p=1}^{n} \frac{2p-1}{2p} = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$ and $\frac{1}{2\sqrt{n}} = \frac{1}{2\sqrt{2}}$. Notice that $6\sqrt{2} > 6 \cdot 1.4 = 8.6 > 8$. So $\frac{6}{8} > 1 > \frac{1}{\sqrt{2}}$, and therefore $\frac{3}{8} > 1 > \frac{1}{2\sqrt{2}}$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n} \frac{2p-1}{2p} > \frac{1}{2\sqrt{n}}$ for $n = 2, \ldots, k$.

Inductive Step: [The following step is completely unmotivated. It was found by working backwards from the desired conclusion.] We know that $(2k+1)^2 = 4k^2 + 4k + 1 > 4k^2 + 4k = 2k(2k+2)$.

Taking the square root of both sides, we get $2k + 1 > \sqrt{2k}\sqrt{2k+2}$. Dividing both sides by $2k + 2$, we get $\frac{2k+1}{2k+2} > \frac{\sqrt{2k}}{\sqrt{2k+2}} = \frac{\sqrt{k}}{\sqrt{k+1}}$.

Now, consider $\prod_{p=1}^{k+1} \frac{2p-1}{2p}$. Using the inductive hypothesis, we have

$\prod_{p=1}^{k+1} \frac{2p-1}{2p} = \frac{2k+1}{2k+2}(\prod_{p=1}^{k} \frac{2p-1}{2p}) > \frac{2k+1}{2k+2}\frac{1}{2\sqrt{k}}$

But we know from the above that $\frac{2k+1}{2k+2}\frac{1}{2\sqrt{k}} > \frac{\sqrt{k}}{\sqrt{k+1}}\frac{1}{2\sqrt{k}} = \frac{1}{2\sqrt{k+1}}$

So $\prod_{p=1}^{k+1} \frac{2p-1}{2p} > \frac{1}{2\sqrt{k+1}}$ which is what we needed to prove.

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(15 points) Recall the following fact about real numbers

Triangle Inequality: For any real numbers x and y, $|x + y| \leq |x| + |y|$.

Use this fact and (strong) induction to prove the following claim:

Claim: For any real numbers $x_1, x_2, \ldots, x_n$ ($n \geq 2$), $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, the claim is exactly the Triangle Inequality, which we're assuming to hold.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$ for any list of n real numbers $x_1, x_2, \ldots, x_n$, where $2 \leq n \leq k$.

Inductive Step: Let $x_1, x_2, \ldots, x_{k+1}$ be a list of $k + 1$ real numbers.

Using the Triangle Inequality, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq |(x_1 + x_2 + \ldots + x_k)| + |x_{k+1}|$

But, by the inductive hypothesis $|(x_1 + x_2 + \ldots + x_k)| + |x_{k+1}| \leq |x_1| + |x_2| + \ldots + |x_k| \leq |x_{k+1}|$.

Putting these two equations together, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq (|x_1| + |x_2| + \ldots + |x_k|) + |x_{k+1}|$.

So $|x_1 + x_2 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + \ldots + |x_k| + |x_{k+1}|$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{7}{12}$, for any integer $n \geq 2$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{7}{12}$, for $n = 2, 3, \ldots, p$.

Inductive Step: Substituing $n = p+1$ into the summation and then using the inductive hypothesis, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} = (\sum_{k=p+1}^{2p} \frac{1}{k}) + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1})$$

Now, notice that $\frac{1}{2p+1} \geq \frac{1}{2}\frac{1}{p+1}$ and $\frac{1}{2p+2} = \frac{1}{2}\frac{1}{p+1}$. So $\frac{1}{2p+1} + \frac{1}{2p+2} \geq \frac{1}{p+1}$. Therefore $\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1} \geq 0$.

Combining the results of the previous two paragraphs, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12}$$

This is what we needed to show.