CS 173, Fall 2016
Examlet 12, Part A

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Discussion: **Thursday** **2** **3** **4** **5** **Friday 9** **10** **11** **12** **1** **2**

(a) (9 points) Suppose that I have a set of p nodes, labelled 1 through p. How many different graphs can I make with this fixed set of nodes? (Isomorphic graphs with differently labelled nodes count as different for this problem.) Briefly justify your answer.

Solution: There are $\frac{p(p-1)}{2}$ possible edges for the graph. For each one, we can choose to include it or not. So there are $2^{\frac{p(p-1)}{2}}$ different possible graphs.

(b) (6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

> For every computer game g, if g has trendy music or g has an interesting plotline, then g is not cheap.

Solution: There is a computer game g such that g has trendy music or an interesting plotline but g is cheap.

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(a) (9 points) Use proof by contradiction to show that there are no positive integer solutions to the equation $a^2 - 4b = 2$. (You may assume that if k is prime then k divides n if and only if k divides n^2.)

Solution: Suppose not. That is, suppose that there are positive integers a and b such that $a^2 - 4b = 2$.

Since $a^2 - 4b = 2$, $a^2 = 2 + 4b = 2(1 + 2b)$. Since $1 + 2b$ is an integer, a^2 is even. This implies that a is even. So $a = 2n$, where n is an integer.

Substituting $a = 2n$ into $a^2 = 2(1 + 2b)$, we get $4n^2 = 2(1 + 2b)$. So $2n^2 = 1 + 2b$. So $1 + 2b$ is even, but it can't be even since b is an integer.

Since the negated claim led to a contradiction, the original claim must be true.

(b) (6 points) Suppose a car dealer is planning to buy a set of Civics, Accords, and Fits (three kinds of cars). The dealer will buy ten cars in total and can buy any number of each type. How many different choices does he have? The sets are unordered, so three Civics and seven Fits is the same as seven Fits and three Civics.

Solution: Using the formula for combinations with repetition, there are

$$\binom{10+2}{2}$$

choices.

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(a) (9 points) Use proof by contradiction to show that $\sqrt{5} - \sqrt{3} < 1$

Solution: Suppose not. That is, suppose that $\sqrt{5} - \sqrt{3} \geq 1$.

Squaring both sides, we get $5 - 2\sqrt{15} + 3 \geq 1$. So then $7 \geq 2\sqrt{15}$. Squaring both sides, we get $49 \geq 4 \cdot 15 = 60$. But $49 \geq 60$ is false.

Since our negated assumption led to a contraiction, the original claim must be true.

(b) (6 points) Use the binomial theorem to find a closed form for the summation $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$. Make sure it's clear how you used the theorem.

Solution: The binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$.

Setting $x = -1$ and $y = 1$ gives us $(-1+1)^n = \sum_{k=0}^{n} (-1)^k 1^{n-k} \binom{n}{k}$

That is $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

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(a) (9 points) A domino has two ends, each of which may be blank or contain between one and n spots. The two ends may have the same number of spots or different numbers of spots. A double-n domino set contains exactly one of each possible dot combination, where the order of the two ends doesn't matter. For example, a double-two domino set contains $(0, 0)$, $(1, 0)$, $(2, 0)$, $(1, 1)$, $(1, 2)$, and $(2, 2)$. Give a general formula for the number of dominoes in a double-n set, explaining why your formula is correct.

Solution: The double-n set contains $n + 1$ dominos with the same number of spots on both ends.

A domino with dissimilar ends corresponds to a set of two numbers in the range 0 through n. There are $\binom{n+1}{2} = \frac{n(n+1)}{2}$ such sets.

So, in total, there are $(n + 1) + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ dominoes in the set.

(b) (6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every relish r, if r is orange and r is not spicy, then r is pungent.

Solution: There is a relish r, such that r is orange and r is not spicy but r is not pungent.

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(a) (9 points) Use proof by contradiction to show that $\sqrt{3}$ is not rational. Hints: assume that your starting fraction is in lowest terms. Also if k is prime, you can assume that k divides n if and only if k divides n^2.

Solution: Suppose not. That is, suppose that $\sqrt{3}$ is rational.

Since $\sqrt{3}$ is rational, $\sqrt{3} = \frac{a}{b}$ for some integers a and b that have no common factors (lowest terms).

Squaring both sides, we get that $3 = \frac{a^2}{b^2}$ and therefore $b^2 = 3a^2$. So 3 must divide b^2, and therefore 3 must divide b. So $b = 3k$ for some integer k.

Substituting $b = 3k$ into $b^2 = 3a^2$ gives us $9k^2 = 3a^2$, so $3k^2 = a^2$. This means that 3 divides a^2 and therefore 3 divides a. But a and b now share a factor of 3, where we assumed that they have no common factors.

Since the negated claim led to a contradiction, the original claim must be true.

(b) (6 points) Chancellor Wise needs to construct a 13-person blue ribbon panel to find a new mascot, but she needs to decide how many members of the group are faculty, undergraduates, graduate students, and staff. How many ways can she choose the composition of the committee?

Solution: Using the formula for combinations with repetition, we get

$$\binom{13+3}{3}$$

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(a) (9 points) Use proof by contradiction to show that are no rational solutions to the equation $x^3 + x + 1 = 0$. Hint: set up x as a fraction in lowest terms, simplify the equation, and look at which combinations of terms can be even/odd.

Solution: Suppose not. That is, suppose that there is a ration number x such that $x^3 + x + 1 = 0$.

Since x is rational, we can write it as $x = \frac{a}{b}$, where a and b are integers which share no common factor. So then our equation becomes $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0$.

Multiplying the whole thing by b^3, we get $a^3 + ab^2 + b^3 = 0$. There are two cases:

Case 1: a is even. Then the first two terms are even. But then b^3 is even, which means b must be even. So a and b share a common factor (contrary to our assumption).

Case 2: a is odd. Then the first term is odd, so $ab^2 + b^3$ must be odd. But this is impossible, because ab^2 and b^3 must both have the same parity as b.

Since the negated claim led to a contradiction, the original claim must be true.

(b) (6 points) In the polynomial $(2x - 3y)^{20}$, what is the coefficient of the term x^5y^{15}? (Please do not attempt to simplify your formula.)

Solution: According to the Binomial Theorem, this term is $\binom{20}{5}(2x)^5(-3y)^{15}$.

This is equal to $\binom{20}{5}2^5x^5(-3)^{15}y^{15}$.

So the coefficient on this term is $\binom{20}{5}2^5(-3)^{15}$.