CS 173, Spring 2016
Examlet 2, Part A

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(15 points) For any two real numbers x and y, the arithmetic mean is $M(x,y) = \frac{x+y}{2}$ and the harmonic mean is $H(x,y) = \frac{2xy}{x+y}$. Use proof by contrapositive to prove the following claim, using these definitions and your best mathematical style.

For all real numbers x and y ($x \neq -y$), if $x \neq y$, then $H(x,y) \neq M(x,y)$.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for any real numbers x and y ($x \neq -y$), if $H(x,y) = M(x,y)$, then $x = y$.

So let x and y be real numbers such that $x \neq -y$. Suppose that $H(x,y) = M(x,y)$. Using the definitions of M and H, this means that $\frac{2xy}{x+y} = \frac{x+y}{2}$.

Multiplying both sides by $x(x+y)$ gives us $4xy = (x+y)^2$.

So $(x+y)^2 - 4xy = 0$. That is, $x^2 - 2xy + y^2 = 0$. Factoring the lefthand side gives us $(x-y)^2 = 0$. So $x - y = 0$. And therefore $x = y$, which is what we needed to show.

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(15 points) Notice that, for any integer p, $\lfloor p \rfloor = \lfloor p + \frac{1}{2} \rfloor = p$. Using this fact and your best mathematical style, prove the following claim:

For any integer n, if n is odd, then $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$

Solution: Let n be an integer and suppose that n is odd. Since n is odd, we can write $n = 2k + 1$, where k is an integer.

Looking at the left side of our equation, we have $\lfloor \frac{n}{2} \rfloor^2 + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor^2 + \lfloor \frac{2k+1}{2} \rfloor = \lfloor k + \frac{1}{2} \rfloor^2 + \lfloor k + \frac{1}{2} \rfloor = k^2 + k$

On the right side, we have $\frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{(2k+1)^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{4k^2+4k+1}{2} \right\rfloor = \frac{1}{2} \lfloor 2k^2 + 2k + \frac{1}{2} \rfloor = \frac{1}{2}(2k^2 + 2k) = k^2 + k$. (Noting that $2k^2 + 2k$ must be an integer because k is an integer.)

So $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$ and therefore $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$.

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(15 points) Prove the following claim, working directly from the definitions of "remainder" and "divides", and using your best mathematical style.

For all real numbers k, m, n and r ($n \neq 0$), if $r = \text{remainder}(m, n)$, $k \mid n$, and $k \mid r$, then $k \mid m$.

Solution: Let k, m, n and r be real numbers ($n \neq 0$). Suppose that $r = \text{remainder}(m, n)$, $k \mid n$, and $k \mid r$.

By the definition of remainder, $m = nq + r$, where q is some integer. (Also r has to be between 0 and n, but that's not required here.)

By the definition of divides, $n = ks$ and $r = kt$, for some integers s and t. Substituting these into the previous equation, we get

$$m = nq + r = (ks)q + kt = k(sq + t)$$

$sq - t$ is an integer because s, t, and q are integers. So m is the product of k and an integer, which means that $k \mid m$.

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(15 points) Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases. You may use the fact that if $a \mid b$, then $a \mid bc$ for any integers a, b, and c.

For any integer n, $n^4 - n^2$ is divisible by 3.

Solution: Let n be an integer. Notice that $n^4 - n^2 = n^2(n^2 - 1)$.

We can write $n = 3q + r$, where r is the remainder of n divided by 3 and q is an integer. There are three cases.

Case 1: $r = 0$. Then $3 \mid n$, so $3 \mid n^2$. and therefore $3 \mid n^2(n^2 - 1)$.

Case 2: $r = 1$. Then $(n^2 - 1) = (9q^2 + 6q + 1) - 1 = 3(3q^2 + 2q)$ So $3 \mid (n^2 - 1)$ and therefore $3 \mid n^2(n^2 - 1)$.

Case 3: $r = 2$. Then $(n^2 - 1) = (9q^2 + 12q + 4) - 1 = 3(3q^2 + 4q + 1)$ So $3 \mid (n^2 - 1)$ and therefore $3 \mid n^2(n^2 - 1)$.

In all three cases, $3 \mid n^2(n^2 - 1)$. So $n^4 - n^2$ is divisible by 3.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a - b = nk$ for some integer n.

Claim: For all integers a, b, c, d, j and k (j and k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$, then $a + c \equiv b + d \pmod{j}$.

Solution:

Let a, b, c, d, j and k be integers, with j and k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a - b = nk$ for some integer n. Similarly $c \equiv d \pmod{k}$ implies that $c - d = mk$ for some integer m. By the definition of divides, $j \mid k$ implies that $k = pj$ for some integer p.

We can then calculate

$$(a + c) - (b + d) = (a - b) + (c - d) = nk + mk = (n + m)k = (n + m)pj$$

Notice that $(n + m)p$ is an integer, since n, m, and p are integers. So, by the definition of congruence mod k, $a + c \equiv b + d \pmod{j}$.