CS 173, Spring 2016
Examlet 5, Part A

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

1. (10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that g is one-to-one.

   Solution: Let m and n be elements of A. Suppose that $g(m) = g(n)$.

   Since $g(m) = g(n)$, $f(g(m)) = f(g(n))$ by substitution. Since $f(g(x)) = x$ for every $x \in A$, $f(g(m)) = m$ and $f(g(n)) = n$. So $f(g(m)) = f(g(n))$ implies that $m = n$.

   Since $g(m) = g(n)$ implies that $m = n$ for any m and n in A, g is one-to-one, which is what we needed to prove.

2. (5 points) $A = \{0, 1, 4, 9, 16, 25, 36, \ldots\}$, i.e. perfect squares starting with 0.

   $B = \{2, 4, 6, 8, 10, 12, 14, \ldots\}$, i.e. the even numbers starting with 2.

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = 2(\sqrt{n} + 1)$

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1. (10 points) Suppose that $f : A \to B$ and $g : B \to C$ are one-to-one. Prove that $g \circ f$ is one-to-one.

   Solution: Let x and y be elements of A and suppose that $g \circ f(x) = g \circ f(y)$. That is $g(f(x)) = g(f(y))$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$.

   We've shown that $g \circ f(x) = g \circ f(y)$ implies $x = y$ for any x and y in A. So $g \circ f$ is one-to-one.

2. (5 points) What's wrong with this attempt to define $f \circ g$?

   > If $f : A \to B$ and $g : B \to C$ are functions, then $f \circ g$ is the function from A to C defined by $(f \circ g)(x) = f(g(x))$.

   Solution: This is applying the two functions in the wrong order. The domain of $f \circ g$ is stated to be A, so x must be an element of A. But we're applying g first and the inputs to g must come from the set B. So this is secretly assuming some overlap among the three sets, which you can't do in a general definition of composition.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) + y, y + 3)$. Prove that g is onto.

   Solution: Suppose that (a, b) is a pair of integers.

   Consider $c = a - b + 3$. c is an integer, since a and b are integers. Since f is onto, this means there is an integer x such that $f(x) = c$.

   Now, let $y = b - 3$. We can then calculate:

   $$g(x, y) = (f(x) + y, y + 3) = (c + y, (b - 3) + 3) = ((a - b + 3) + (b - 3), b) = (a, b)$$

   So we've found a point (x, y) such that $g(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is increasing (but perhaps not strictly increasing). Dumbledore claims that f must be one-to-one. Is he correct? Briefly explain why he is or give a concrete counter-example.

   Solution: He's wrong. Suppose that $f(n) = 0$ for every input value n. Then f is (non-strictly) increasing, but not one-to-one.

CS 173, Spring 2016
Examlet 5, Part A

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Discussion: Monday 9 10 11 12 1 2 3 4 5

1. (10 points) Let P be the set of pairs of positive integers. Suppose that $f : P \to \mathbb{R}^2$ is defined by $f(x, y) = (\frac{x}{y}, x + y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of P, i.e. pairs of positive integers. Suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(\frac{x}{y}, x + y) = (\frac{p}{q}, p + q)$. So $\frac{x}{y} = \frac{p}{q}$ and $x + y = p + q$.

   Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{py}{q}$. Substituting this into $x + y = p + q$ gives us $\frac{py}{q} + y = p + q$. So $\frac{py+yq}{q} = p + q$. I.e. $\frac{y(p+q)}{q} = p + q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

   Substituting $y = q$ into $x + y = p + q$ gives us $x + y = p + y$, so $x = p$.

   Therefore $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto. Be specific.

   Solution: Let $f(n) = n + 1$. Then f is one-to-one, but 0 isn't in the image of f.

CS 173, Spring 2016
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1. (10 points) If a is any real number, (a, ∞) is the set of all real numbers greater than a. Let's define the function $f : (0, \infty) \to \left(\frac{1}{3}, \infty\right)$ by $f(x) = \dfrac{x^2+2}{3x^2}$. Prove that f is onto.

   Solution: Let $y \in \left(\frac{1}{3}, \infty\right)$. Then $y > \frac{1}{3}$, so $3y > 1$, and therefore $3y - 1 > 0$.

   So $\frac{2}{3y-1}$ is defined and positive. So consider $x = \sqrt{\frac{2}{3y-1}}$. x is defined and belongs to $(0, \infty)$.

   Then $x^2 = \frac{2}{3y-1}$. So $x^2 + 2 = \frac{2}{3y-1} + 2 = \frac{2+(6y-2)}{3y-1} = \frac{6y}{3y-1}$. And $3x^2 = \frac{6}{3y-1}$.

   Then $f(x) = \frac{x^2+2}{3x^2} = \frac{6y}{6} = y$.

   So we've found a pre-image for our original value y, which is what we needed to do.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in M, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that f is onto.

   Solution: Let m be an element of A. We need to find a pre-image for m.

   Consider $n = g(m)$. n is an element of B. Furthermore, since $f(g(x)) = x$ for every $x \in A$, we have $f(n) = f(g(m)) = m$.

   So n is a pre-image of m.

   Since we can find a pre-image for an arbitrarily chosen element of A, f is onto.

2. (5 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ are functions. Let's define the function $f + g$ by $(f + g)(x) = f(x) + g(x)$. Adele claims that if f and g are one-to-one, then $f + g$ is one-to-one. Is this correct? Briefly explain why it is or give a counter-example.

   Solution: This is not correct. Suppose that $g(x) = -f(x)$ for every input x. Then $(f + g)(x) = 0$ for any x, making $f + g$ not one-to-one.