CS 173, Spring 2016
Examlet 8, Part A

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(20 points) Suppose that $g : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$g(1) = 2$

$g(2) = 8$

$g(n) = 4(\ g(n-1) - g(n-2)\ )$

Use (strong) induction to prove that $g(n) = n2^n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $g(n) = 2$ and $n2^n = 1 \cdot 2^1 = 2$, so the claim holds. At $n = 2$, $g(n) = 8$ and $n2^n = 2 \cdot 2^2 = 8$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = n2^n$ for $n = 1, \ldots, k-1$.

Rest of the inductive step: In particular, the inductive hypothesis tells us that $g(k-1) = (k-1)2^{k-1}$ and $g(k-2) = (k-2)2^{k-2}$

So

$$\begin{aligned} g(k) &= 4(g(k-1) - g(k-2)) = 4((k-1)2^{k-1} - (k-2)2^{k-2} \\ &= 4(k-1)2^{k-1} - 4(k-2)2^{k-2} = (2k-2)2^k - (k-2)2^k \\ &= k2^k \end{aligned}$$

So $g(k) = k2^k$, which is what we needed to prove.

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(20 points) Suppose that $g : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$$g(1) = 1$$

$$g(2) = 8$$

$$g(n) = g(n-1) + 2g(n-2)$$

Use (strong) induction to prove that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $g(n) = 1$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^0 + 2(-1) = 3 - 2 = 1$, so the claim holds.

At $n = 2$, $g(n) = 8$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n = 1, \ldots, k - l$.

Rest of the inductive step: In particular, by the inductive hypothesis, $g(k-1) = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$ and $g(k-2) = 3 \cdot 2^{k-3} + 2(-1)^{k-2}$.

So

$$\begin{aligned}
g(k) &= g(k-1) - 2g(k-2) = (3 \cdot 2^{k-2} + 2(-1)^{k-1}) + 2(3 \cdot 2^{k-3} + 2(-1)^{k-2}) \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 6 \cdot 2^{k-3} + 4(-1)^{k-2} \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 3 \cdot 2^{k-2} - 4(-1)^{k-1} \\
&= 6 \cdot 2^{k-2} - 2(-1)^{k-1} = 3 \cdot 2^{k-1} + 2(-1)^k
\end{aligned}$$

So $g(k) = 3 \cdot 2^{k-1} + 2(-1)^k$, which is what we needed to show.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is a function such that

$f(1)$ and $f(2)$ are odd

$f(n) = 2f(n-1) + 3f(n-2)$

Use (strong) induction and the definition of odd to prove that $f(n)$ is odd for all positive integers n.

Solution: Proof by induction on n.

Base case(s): We're told in so many words that $f(1)$ and $f(2)$ are odd. So the claim is true at $n = 1$ and $n = 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n)$ is odd for $n = 1, \ldots, k-1$.

Rest of the inductive step: In particular, by the inductive hypothesis, $f(k-1)$ and $f(k-2)$ are odd. So $f(k-1) = 2p+1$ and $f(k-2) = 2q+1$, for some integers p and q.

$$\begin{aligned} f(k) &= 2f(k-1) + 3f(k-2) = 2(2p+1) + 3(2q+1) \\ &= 4p + 2 + 6q + 3 = 4p + 6q + 5 = 2(2p + 3q + 2) + 1 \end{aligned}$$

Let $t = 2p + 3q + 2$. Then $f(k) = 2t + 1$ and tt is an integer since p and q are integers. So $f(k)$ is odd, which is what we needed to show.

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(20 points) Recall that

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Use (strong) induction to prove that $(\cos x + i \sin x)^n = \cos(nx) + i\sin(nx)$, for any natural number n. (i is the square root of -1.)

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $(\cos x + i\sin x)^0 = 1$ and $\cos(0) + i\sin(0) = 1 + 0 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(\cos x + i \sin x)^n = \cos(nx) + i\sin(nx)$, for $n = 0, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $(\cos x + i\sin x)^k = \cos(kx) + i\sin(kx)$.

Then

$$\begin{aligned}
(\cos x + i\sin x)^{k+1} &= (\cos x + i\sin x)(\cos x + i \sin x)^k \\
&= (\cos x + i\sin x)(\cos kx + i\sin kx) \\
&= \cos kx\cos x - \sin kx \sin x + i(\cos kx \sin x + \sin kx \cos x) \\
&= \cos((k+1)x) + i\sin((k+1)x)
\end{aligned}$$

So $(\cos x + i\sin x)^{k+1} = \cos((k+1)x) + i\sin((k+1)x)$, which is what we needed to show.

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(20 points) Suppose that $g : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$$g(1) = g(2) = 1$$

$$g(n) = 2g(n-1) + 3g(n-2)$$

Use (strong) induction to prove that $g(n) = \frac{1}{2}(3^{n-1} - (-1)^n)$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $g(n) = 1$ and $\frac{1}{2}(3^{n-1} - (-1)^n) = \frac{1}{2}(1 - (-1)) = 1$ $n2^n = 1 \cdot 2^1 = 2$, so the claim holds. At $n = 2$, $g(n) = 1$ and $\frac{1}{2}(3^{n-1} - (-1)^n) = \frac{1}{2}(3-1) = 1$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = \frac{1}{2}(3^{n-1} - (-1)^n)$ for $n = 1, \ldots, k-1$.

Rest of the inductive step: In particular, the inductive hypothesis tells us that $g(k-1) = \frac{1}{2}(3^{k-2} - (-1)^{k-1})$ and $g(k-2) = \frac{1}{2}(3^{k-3} - (-1)^{k-2})$

So

$$\begin{aligned} g(k) &= 2(g(k-1) + 3g(k-2)) = 2(\frac{1}{2}(3^{k-2} - (-1)^{k-1})) + 3(\frac{1}{2}(3^{k-3} - (-1)^{k-2})) \\ &= \frac{1}{2}(2 \cdot 3^{k-2} - 2(-1)^{k-1} + 3 \cdot 3^{k-3} - 3(-1)^{k-2}) \\ &= \frac{1}{2}(2 \cdot 3^{k-2} - 2(-1)^{k-1} + 3^{k-2} + 3(-1)^{k-1}) \\ &= \frac{1}{2}(3 \cdot 3^{k-2} + (-1)^{k-1} = \frac{1}{2}(3^{k-1} - (-1)^k) \end{aligned}$$

So $g(k) = \frac{1}{2}(3^{k-1} - (-1)^k)$, which is what we needed to prove.

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(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$.

Use (strong) induction to prove that $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.

Solution: Proof by induction on n.

Base case(s): At $n = 2$, $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = (1 - \frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

Rest of the inductive step: In particular, from the inductive hypothesis $\prod_{p=2}^{k}(1 - \frac{1}{p^2}) = \frac{k+1}{2k}$.

So

$$\begin{aligned}
\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) &= (\prod_{p=2}^{k}(1 - \frac{1}{p^2}))(1 - \frac{1}{(k+1)^2}) \\
&= (\frac{k+1}{2k})(1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2} \\
&= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2}{2k(k+1)} - \frac{1}{2k(k+1)} \\
&= \frac{(k+1)^2 - 1)}{2k(k+1)} = \frac{k^2 + 2k)}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)}
\end{aligned}$$

So $\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.