CS 173, Spring 2016
Examlet 8, Part B

NETID:

FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= 17 \\ F(n) &= 3F(n/2), \text{ for } n \geq 4 \end{aligned}$$

Use unrolling to find the closed form for F. Show your work and simplify your answer.

Solution:

$$\begin{aligned} F(n) &= 3F(n/2) = 3(3F(n/4)) = 3(3(3(F(n/2^3)))) \\ &= 3^3F(n/2^3) \\ &= 3^kF(n/2^k) \end{aligned}$$

We'll hit the base case when $n/2^k = 2$, i.e. $n = 2^{k+1}$, i.e. $k + 1 = \log_2 n$, $k = \log_2 n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} F(n) &= 3^{\log_2 n - 1} \cdot 17 \\ &= 17/3 \cdot 3^{\log_2 n} = 17/3 \cdot 3^{\log_3 n \log_2 3} \\ &= 17/3n^{\log_2 3} \end{aligned}$$

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1. (8 points) Suppose we have a function g defined by

$$\begin{aligned} g(0) &= g(1) = c \\ g(n) &= kg(n-2) + n^2, \text{ for } n \geq 2 \end{aligned}$$

where k and c are constants. Express $g(n)$ in terms of $g(n-6)$ (where $n \geq 6$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} g(n) &= kg(n-2) + n^2 \\ &= k(kg(n-4) + (n-2)^2) + n^2 \\ &= k(k(kg(n-6) + (n-4)^2) + (n-2)^2) + n^2 \\ &= k^3 g(n-6) + k^2(n-4)^2 + k(n-2)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

Shorthand for the n-dimensional hypercube. C_n ☐ H_n ☐ Z_n ☐ Q_n ☑ (√)

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1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} f(n) &= 3f(n-1) + n^2 \\ &= 3(3f(n-2) + (n-1)^2) + n^2 \\ &= 3(3(3f(n-3) + (n-2)^2) + (n-1)^2) + n^2 \\ &= 27f(n-3) + 9(n-2)^2 + 3(n-1)^2 + n^2 \end{aligned}$$

2. (2 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is such that $f(n) = n^2$. Give a recursive definition of f

Solution:

$f(0) = 0$, and $f(n+1) = f(n) + 2n + 1$ for $n \geq 0$.

You could also have used $f(n) = f(n-1) + 2n - 1$ for $n \geq 1$.

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(10 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant.

Your partner has already figured out that

$$f(n) = 5^k f(n-2k) + \sum_{p=0}^{k-1} d5^p$$

Finish finding the closed form for $f(n)$ assuming that n is even. Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

Solution:

To find the value of k at the base case, set $n - 2k = 0$. Then $n = 2k$, so $k = n/2$. Substituting this into the above, we get

$$\begin{aligned} f(n) &= 5^k f(n-2k) + \sum_{p=0}^{k-1} d5^p \\ &= 5^{n/2} \cdot 3 + d \sum_{p=0}^{n/2-1} 5^p \\ &= 3 \cdot 5^{n/2} + d(\frac{5^{n/2}-1}{4}) \\ &= (3 + d/4)5^{n/2} + d/4 = (3 + d/4)(\sqrt{5})^n + d/4 \end{aligned}$$

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1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant. Express $f(n)$ in terms of $f(n-6)$ (where $n \geq 6$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} f(n) &= 5f(n-2) + d \\ &= 5(5(f(n-4) + d) + d) \\ &= 5(5(5(f(n-6) + d) + d) + d) \\ &= 5^3 f(n-6) + (25 + 5 + 1)d \\ &= 5^3 f(n-6) + 31d \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

$f(n) = n!$ can be defined recursively by $f(0) = 1$, and $f(n) = nf(n-1)$ for all integers ...	$n \geq 0$ ☐	$n \geq 1$ ☑ (√)	$n \geq 2$ ☐

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(10 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = n^2$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^{\log_2 n} \cdot c + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 4^{\log_2 n} \cdot c + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= cn^2 + n(2^{\log_2 n} - 1) = cn^2 + n^2 - n = (c+1)n^2 - n \end{aligned}$$