CS 173, Spring 2016
Examlet 9, Part A

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FIRST: **LAST:**

Discussion: Monday 9 10 11 12 1 2 3 4 5

(18 points) Suppose that a chocolate bar is a rectangle divided into a p by q grid of squares. Breaking a chocolate bar along one of the grid lines divides it into two smaller rectangular bars. Prove that it takes $pq - 1$ breaks to divide a chocolate bar into individual squares, for any choice of the order in which you choose break lines.

Solution: The induction variable is named **h** and it is the **area** of/in the bar.

Base Case(s): At h=1, the bar already consists of a single square. So we don't need to break it up further. That is, we need 0, i.e. h-1 breaks to divide it up. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Any chocolate bar of area h can be divided into individual squares using h-1 breaks, for n from 1 up through k-1.

Inductive Step: Suppose that B is a chocolate bar of area k, where $k > 1$. Let's break B along any grid line, creating two bars X and Y.

Let x and y be the areas of X and Y. Then $x + y = k$.

By the inductive hypothesis, we can reduce X to individual squares using $x - 1$ breaks. Similarly, we can reduce Y to individual squares using $y - 1$ breaks.

Therefore, to reduce B to individual squares, we use our initial break, then break up X and Y using $x - 1$ and $y - 1$ breaks. So the total number of breaks required to divide up B is $1 + (x - 1) + (y - 1) = x + y - 1 = k - 1$. So breaking up B requires $k - 1$ breaks, which is what we needed to prove.

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(18 points) A Chicago tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 5, 17, or 23.
- A node with one child contains the same number as its child.
- A node with two children contains the value $x(y+1)$, where x and y are the values in its children.

Use strong induction to prove that the value in the root of a Chicago tree is always positive.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The smallest Chicago trees consist of a single root node, which is also a leaf. By the definition of Chicago tree, this must contain 5, 17, or 23, all of which are positive.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the root of a Chicago tree of height h is always positive, for $h = 0, \ldots, k-1$.

Inductive Step: Let T be a Chicago tree of height k. There are two cases for what the top of T looks like.

Case 1: T consists of a root r with a single subtree S under it. r contains the same number as the root of S. Since S must be shorter than k, its root contains a positive number by the inductive hypothesis. Since r has the same label, r contains a positive number.

Case 2: T consists of a root r with a two subtrees S_1 and S_2. Suppose that the roots of S_1 and S_2 contain the numbers x and y. Then, by the definition of Chicago tree, r contains $x(y+1)$.

Since S_1 and S_2 are shorter than k, x and y must be positive by the inductive hypothesis. Since y is positive, so is $y+1$. Since x and $y+1$ are positive, so is $x(y+1)$. So the root of T contains a positive number.

So, in either case, the root of T contains a positive number.

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(18 points) A Boston tree is a binary tree whose nodes are labelled with natural numbers such that

- If v is a leaf node, then v has label 15, 30, 20, or 40.
- If v has one child with label x, then v has label $3x$.
- If v has two children with labels x and y, then v has label $\gcd(x, y)$, i.e. the greatest common divisor of x and y. E.g. if the children have labels 30 and 40, the parent has label 10.

Use (strong) induction to prove that the root node of every Boston tree is divisible by 5. Assume we are all familiar with basic facts about divisibility, so you don't need to go all the way back to the definition.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): At $h = 0$, the Boston tree consists of a single node with label 15, 30, 20, or 40. All of these are divisible by 5, so the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the root of every Boston tree is divisble by 5, for $h = 0, 1, \ldots, k-1$.

Inductive Step: Let T be a Boston tree of height $k > 0$, with root node v.

Case 1: v has one child, with label x. By the induction hypothesis, x is divisible by 5. Then v has label $3x$, which must also be divisible by 5.

Case 2: v has two children with labels x and y. By the induction hypothesis, x and y are both divisible by 5. Then $\gcd(x, y)$, is also divisible by 5. Since this is the label on v, the label on v is divisible by 5.

In both cases, the label on the root of T is divisible by 5, which is what we needed to show.

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(18 points) Suppose grammar G has start symbol S, terminal symbols a, b, c, and d, and these rules:

$$S \rightarrow S\ a\ b\ S \ \mid\ S\ d\ S \ \mid\ c\ a \ \mid\ b\ a$$

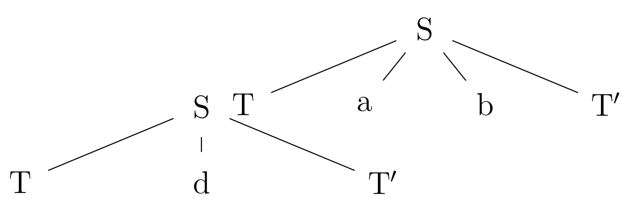
For any tree T, define $A(T)$ to be the number of nodes with label a in T, and similarly for $B(T)$ and $C(T)$. Use (strong) induction to prove that $A(T) = B(T) + C(T)$ for any tree T matching grammar G.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): At $h = 1$, $A(T) = B(T) + C(T)$ holds for both of the trees (shown below).

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: For any tree T matching grammar G of height $h = 1, \ldots, k-1$, $A(T) = B(T) + C(T)$.

Inductive Step: Consider a tree P matching grammar G of height $k > 1$. P must have one of the following two forms, where T and T' are trees matching grammar G.



By the induction hypothesis, $A(T) = B(T) + C(T)$ and $A(T') = B(T') + C(T')$.

In the lefthand case, we have $A(P) = A(T) + A(T') + 1$, $B(P) = B(T) + B(T') + 1$, and $C(P) = C(T) + C(T')$. Substituting, we get

$$\begin{aligned} A(P) &= A(T) + A(T') + 1 = B(T) + C(T) + B(T') + C(T') + 1 \\ &= (B(T) + B(T') + 1) + (C(T) + C(T')) = B(P) + C(P) \end{aligned}$$

In the righthand case, $A(P) = A(T) + A(T')$, $B(P) = B(T) + B(T')$, and $C(P) = C(T) + C(T')$. So $A(P) = A(T) + A(T') = B(T) + C(T) + B(T') + C(T') = (B(T) + B(T')) + (C(T) + C(T')) = B(P) + C(P)$

In both cases, $A(P) = B(P) + C(P)$, which is what we needed to show.

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(18 points) A Memphis tree is a binary tree containing 2D points such that:

- Each leaf node contains $(3, 1)$, $(-2, -5)$, or $(2, 2)$.
- An internal node with one child labelled (a, b) has label $(a + 1, b - 1)$.
- An internal node with two childen labelled (x, y) and (a, b) has label $(x + a, y + b)$.

Use (strong) induction to prove that the point in the root node of any Memphis tree is on or below the line $x = y$.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest Memphis trees consist of a single node containing $(3, 1)$, $(-2, -5)$, or $(2, 2)$. All three of these points lie on or below the line $x = y$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the point in the root node of any Memphis tree is on or below the line $x = y$, for trees of height $h = 0, 1, \ldots, k - 1$. $(k \geq 1)$.

Inductive Step: Let T be a Memphis tree of height k. There are two cases.

Case 1: The root of T has one child subtree, whose root contains (a, b). By the inductive hypothesis, (a, b) is on or below $x = y$, i.e. $b \leq a$. By the definition of a Memphis tree, the root of T contains $(a + 1, b - 1)$. Since $b \leq a$, $b - 1 \leq a + 1$, so this point is on or below $x = y$.

Case 2: The root of T has two child subtrees, whose roots contain (x, y) and (a, b). Then the root of T contains $(x + a, y + b)$. By the inductive hypothesis, $y \leq x$ and $b \leq a$, so $y + b \leq x + a$. So $(x + a, y + b)$ is on or below $x = y$.

In both cases the root note contains a point on or below $x = y$, which is what we needed to show.

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(18 points) Fremont trees are binary trees whose nodes are labelled with strings, such that

- Each leaf node has label `tip`, `top`, or `tack`
- If a node has one child, it has label $\alpha\alpha$ where where α is the child's label. E.g. if the child has label `top` then the parent has `toptop`.
- If a node has two children, it contains $\alpha\beta$ where α and β are the child labels. E.g. if the children have labels `tip` and `top`, then the parent has label `tiptop`.

Let S(n) be the length of the label on node n. Let L(n) be the number of leaves in the subtree rooted at n. Use (strong) induction to prove that $S(n) \geq 3L(n)$ if n is the root node of any Fremont tree.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base case(s): $h = 0$. The tree consists of a single leaf node, so $L(n) = 1$. The node has label `tip`, `top`, or `tack`, so $S(n) \geq 3$. So $S(n) \geq 3L(n)$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $S(n) \geq 3L(n)$ if n is the root node of any Fremont tree of height $< k$ (where $k \geq 1$).

Rest of the inductive step:

Suppose that T is a Fremont tree of height k. There are two cases:

Case 1: The root n of T has a single child node p. By the inductive hypothesis $S(p) \geq 3L(p)$. $L(n) = L(p)$. And $S(n) = 2S(p) \geq 6L(p) = 6L(n) \geq 3L(n)$. So $S(n) \geq 3L(n)$.

Case 2: The root n of T has two children p and q. By the inductive hypothesis $S(p) \geq 3L(p)$ and $S(q) \geq 3L(q)$.

Notice that $L(n) = L(p) + L(q)$. And $S(n) = S(p) + S(q)$.

So $S(n) = S(p) + S(q) \geq 3L(p) + 3L(q) = 3L(n)$.