CS 173, Spring 2016
Examlet 10, Part A

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Discussion: Monday 9 10 11 12 1 2 3 4 5

(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \leq 2\sqrt{n}$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1 \leq 2 = 2 \cdot n$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \leq 2\sqrt{n}$ for $n = 1, 2, \ldots, k$.

Inductive Step:

First, notice that $(\sqrt{k} - \sqrt{k+1}\ )^2 \geq 0$. Multiplying this out gives us $k - 2\sqrt{k}\sqrt{k+1} + (k+1) \geq 0$. So $2k + 1 \geq 2\sqrt{k}\sqrt{k+1}$. ¿ Using this inequality plus the inductive hypothesis, we can compute

$$
\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} &= \left(\sum_{p=1}^{k} \frac{1}{\sqrt{p}}\right) + \frac{1}{\sqrt{k+1}} \\
&\leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{2\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \\
&\leq \frac{(2k+1)+1}{\sqrt{k+1}} = \frac{2k+2}{\sqrt{k+1}} = \frac{2(k+1)}{\sqrt{k+1}} = 2\sqrt{k+1}
\end{aligned}
$$

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(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by

$f(1) = f(2) = 1$

$f(n) = \frac{1}{2}f(n-1) + \frac{1}{f(n-2)}$

Use (strong) induction to prove that $1 \leq f(n) \leq 2$ for all positive integers n.

Hint: prove both inequalities together using one induction.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ and $n = 2$, $f(n) = 1$. So $1 \leq f(n) \leq 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $1 \leq f(n) \leq 2$ for $n = 1, 2, \ldots k-1$.

Inductive Step: From the inductive hypothesis, we know that $1 \leq f(k-1) \leq 2$ and $1 \leq f(k-2) \leq 2$.

So $\frac{1}{2} \leq \frac{1}{2}f(k-1) \leq \frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \leq \frac{1}{f(k-2)} \leq \frac{1}{1} = 1$.

Using the upper bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \leq 1 + 1 = 2$.

Using the lower bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \geq \frac{1}{2} + \frac{1}{2} = 1$.

So $1 \leq f(k) \leq 2$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=2}^{n} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{n}$ for all integers $n \geq 2$

Solution: Proof by induction on n.

Base Case(s): At $n = 2$, $\sum_{p=2}^{n} \frac{1}{p^2} = \frac{1}{4} \leq \frac{3}{4} - \frac{1}{2}$. So the claim holds at $n = 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=2}^{n} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{n}$ for $n = 2, 3, \ldots, k$

Inductive Step: Notice that $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}$.

So $-\frac{1}{k} + \frac{1}{(k+1)^2} \leq -\frac{1}{(k+1)}$.

So $\frac{3}{4} - \frac{1}{k} + \frac{1}{(k+1)^2} \leq \frac{3}{4} - \frac{1}{(k+1)}$.

By the inductive hypothesis, we know that $\sum_{p=2}^{k} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{k}$. Using this fact and the above work, we can compute:

$\sum_{p=2}^{k+1} \frac{1}{p^2} = \sum_{p=2}^{k} \frac{1}{p^2} + \frac{1}{(k+1)^2} \leq (\frac{3}{4} - \frac{1}{k}) + \frac{1}{(k+1)^2} \leq \frac{3}{4} - \frac{1}{(k+1)}$

So $\sum_{p=2}^{k+1} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{k+1}$, which is what we needed to show.

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(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by

$f(1) = 2$

$f(n) = \frac{1}{2}f(n-1) + \frac{5}{2f(n-1)}$

Use (strong) induction to prove that $2 \leq f(n) \leq \frac{5}{2}$ for any positive integer n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $f(1) = 2$. So clearly $2 \leq f(1) \leq \frac{5}{2}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $2 \leq f(n) \leq \frac{5}{2}$, for $n = 1, 2, \ldots, k-1$.

Inductive Step: From the inductive hypothesis, we know that $2 \leq f(k-1) \leq \frac{5}{2}$.

So $1 \leq \frac{1}{2}f(k-1) \leq \frac{5}{4}$.

And $\frac{2}{5} \leq \frac{1}{f(k-1)} \leq \frac{1}{2}$ So $1 = \frac{2}{5} \cdot \frac{5}{2} \leq \frac{5}{2f(k-1)} \leq \frac{5}{2} \cdot \frac{1}{2} = \frac{5}{4}$

So $2 \leq \frac{1}{2}f(k-1) + \frac{5}{2f(k-1)} \leq \frac{5}{2}$.

Thus $2 \leq f(k) \leq \frac{5}{2}$, which is what we needed to show.

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(15 points) Suppose that $0 < q < \frac{1}{2}$. Use (strong) induction to prove the following claim:

Claim: $(1+q)^n \leq 1 + 2^n q$, for all positive integers n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(1+q)^n = 1 + q$ Also $1 + 2^n q = 1 + 2q$. So $(1+q)^n \leq 1 + 2^n q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(1+q)^n \leq 1+2^n q$, for $n = 1, 2, \ldots, k$.

Inductive Step: From the inductive hypothesis, we know that $(1+q)^k \leq 1 + 2^k q$.

At $n = k+1$, we have

$$\begin{aligned} (1+q)^{k+1} &= (1+q)(1+q)^k \leq (1+q)(1+2^k q) \\ &= 1 + q + 2^k q + 2^k q^2 = 1 + q(1 + 2^k + 2^k q) \end{aligned}$$

Recall that $q < \frac{1}{2}$, so $2^k q < 2^{k-1}$. Also notice that $1 \leq 2^{k-1}$. Using these facts, we get

$$(1+q)^{k+1} \leq \; = 1 + q(1 + 2^k + 2^k q) \leq 1 + q(2^{k-1} + 2^k + 2^{k-1}) = 1 + 2^{k+1} q$$

So $(1+q)^{k+1} \leq 1 + 2^{k+1} q$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim. You may use the fact that $\sqrt{2} \le 1.5$.

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \ge 2\sqrt{n+1} - 2$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$. Also $2\sqrt{n+1} - 2 = 2\sqrt{2} - 2 \le 2 \cdot 1.5 - 2 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \ge 2\sqrt{n+1} - 2$ for $n = 1, 2, \ldots, k$.

Inductive Step: First, notice that $(\sqrt{k+1} - \sqrt{k+2}\,)^2 \ge 0$. Multiplying this out gives us $(k+1) - 2\sqrt{k+1}\sqrt{k+2} + (k+2) \ge 0$. So $2k+3 \ge 2\sqrt{k+1}\sqrt{k+2}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \ge 2\sqrt{k+1} - 2$. So then

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} &= \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \ \ge\ \frac{1}{\sqrt{k+1}} + 2\sqrt{k+1} - 2 \\
&= \frac{1}{\sqrt{k+1}} + \frac{2(k+1)}{\sqrt{k+1}} - 2 = \frac{1+2(k+1)}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2 \\
&\ge \frac{2\sqrt{k+1}\sqrt{k+2}}{\sqrt{k+1}} - 2 = 2\sqrt{k+2} - 2
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \ge 2\sqrt{k+2} - 2$, which is what we needed to show.