CS 173, Fall 17 Examlet 2, Part A

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(15 points) Prove the following claim, using direct proof and your best mathematical style.

For any integers x, y, and z, if $100x + 10y + z$ is divisible by 9, then $x + y + z$ is divisible by 9.

Hint: analyze the difference between $100x + 10y + z$ and $x + y + z$.

Solution: Let a, b, and c be integers and suppose that $100x + 10y + z$ is divisible by 9.

By the definition of divides, $100x + 10y + z = 9k$, where k is an integer.

Notice that $100x + 10y + z = (99x + 9y) + (x + y + z)$. So $(x + y + z) = (100x + 10y + z) - (99x + 9y)$. Substituting $100x+10y+z = 9k$ into this, we get $(x+y+z) = 9k-(99x+9y)$. So $(x+y+z) = 9(k-11x-y)$.

Let $m = k - 11x - y$. m is an integer because k, x, and y are integers. So $(x + y + z) = 9m$, where m is an integer. So $(x + y + z)$ is divisible by 9.

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(15 points) Prove the following claim, working directly from the definitions of "remainder" and "divides", and using your best mathematical style.

For all real numbers k, m, n and r ($n \neq 0$), if $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$, then $k \mid r$.

Solution: Let k, m, n and r be real numbers ($n \neq 0$). Suppose that $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$.

By the definition of remainder, $m = nq + r$, where q is some integer. (Also r has to be between 0 and n, but that's not required here.) So $r = m - nq$.

By the definition of divides, $m = ks$ and $n = kt$, for some integers s and t. Substituting these into the previous equation, we get

$$r = m - nq = ks - ktq = k(s - tq)$$

$s - tq$ is an integer because s, t, and q are integers. So r is the product of k and an integer, which means that $k \mid r$.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all real numbers x and y, $x \neq 0$, if x and $\frac{y+1}{2}$ are rational, then $\frac{5}{x} + y$ is rational.

Solution: Let x and y be real numbers, where $x \neq 0$. Suppose that x and $\frac{y+1}{2}$ are rational.

By the definition of rational, $x = \frac{m}{n}$ and $\frac{y+1}{2} = \frac{p}{q}$, where m, n, p, and q are rationals, n and q non-zero. Since x is non-zero, m is also non-zero.

Since $x = \frac{m}{n}$ and x is not zero, $\frac{5}{x} = \frac{5n}{m}$.

Since $\frac{y+1}{2} = \frac{p}{q}$, $y + 1 = \frac{2p}{q}$. So $y = \frac{2p}{q} - 1 = \frac{2p-q}{q}$.

Combining these, we get that $\frac{5}{x} + y = \frac{5n}{m} + \frac{2p-q}{q} = \frac{5nq+2pm-qm}{mq}$. $5nq + 2pm - qm$ and mq are integers, since n, m, p, and q are integers. mq can't be zero because m and q are both non-zero. So $\frac{5}{x} + y$ is the ratio of two integers and therefore rational.

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(15 points) Recall that $\gcd(m,n)$ is the largest integer that divides both m and n. Use this definition, the definition of divides, and your best mathematical style to prove the following claim by contrapositive.

For all integers p and q, if $p + 6q = 23$ then $\gcd(p,q) \neq 7$.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all integers p and q, if $\gcd(p,q) = 7$, then $p + 6q \neq 23$.

Let p and q be integers and suppose that $\gcd(p,q) = 7$. Then $7 \mid p$ and $7 \mid q$ by the definition of gcd. By the definition of divides, this implies that $p = 7m$ and $q = 7n$, for some integers m and n.

So $p + 6q = 7m + 6(7n) = 7(m + 6n)$. This mean that $p + 6q$ is divisible by 7. Since we know that 23 isn't divisible by 7, $p + 6q$ can't be equal to 23.

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(15 points) An integer k is a perfect square if $k = n^2$ where n is a non-negative integer. Prove the following claim:

For any integer p, if $p \geq 8$ and $p + 1$ is a perfect square, then p is composite (aka not prime).

Solution: Let p be an integer and suppose that $p \geq 8$ and $p + 1$ is a perfect square.

By the definition of a perfect square, $p + 1 = n^2$ where n is a non-negative integer. Then $p = n^2 - 1 = (n + 1)(n - 1)$.

Since $p \geq 8$, $n^2 = p+1 \geq 9 \geq 3^2$. Since n isn't negative, We must have $n \geq 3$. So then $n+1 \geq n-1 \geq 2$.

So we can factor p into two integers $n + 1$ and $n - 1$, neither of which can be one. So p is composite.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

Claim: For all integers a, b, c, d, and k (k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $a^2 + c \equiv b^2 + d \pmod{k}$.

Solution:

Let a, b, c, d, and k be integers, with k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a = b + nk$ for some integer n. Similarly, $c \equiv d \pmod{k}$ implies that $c = d + mk$ for some integer m. Then we can calculate

$$a^2 + c = (b + nk)^2 + (d + mk) = b^2 + 2bnk + n^2k^2 + d + mk = b^2 + d + k(2bn + n^2k + m)$$

If we let $p = 2bn + n^2k + m$, then we have $a^2 + c = (b^2 + d) + kp$. Also, p must be an integer since b, n, k, and m are integers. So, by the definition of congruence mod k, $a^2 + c \equiv b^2 + d \pmod{k}$.