CS 173, Fall 17 Examlet 4, Part A

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(a, b)T(p, q)$ if and only if $ab \mid p$

Working directly from the definition of divides, prove that T is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $ab \mid p$ and $pq \mid m$.

By the definition of divides, we then have $abx = p$ and $pqy = m$, for some integers x and y. Substituting the first equation into the second, we get $(abx)qy = m$. That is $(ab)(xqy) = m$. Since x, y, and q are all integers, so is xqy. So this implies that $ab \mid m$. So $(a, b)T(m, n)$, which is what we needed to show.

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Let's define a relation R on $\mathbb{Z}^3$ as follows:

$(a, b, c)R(x, y, z)$ if and only if $c = x$, $a = y$, and $b = z$.

Working directly from this definition, prove that R is antisymmetric.

Solution: Let (a, b, c) and (x, y, z) be triples of integers. Suppose that $(a, b, c)R(x, y, z)$ and $(x, y, z)R(a, b, c)$.

By the definition of R, $(a, b, c)R(x, y, z)$ implies that $c = x$, $a = y$, and $b = z$.

Alos by the definition of R, $(x, y, z)R(a, b, c)$ implies $z = a$, $x = b$, and $y = c$.

Chaining these equalities together, we get

$a = y = c = x = b = z = a$

So all six integers must be equal. In particular, $a = x$, $b = y$, and $c = z$. So $(a, b, c) = (x, y, z)$.

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Define the relation $\sim$ on $\mathbb{Z}$ by

$x \sim y$ if and only if $5 \mid (3x + 7y)$

Working directly from the definition of divides, prove that $\sim$ is transitive.

Solution: Let x, y, and z be integers. Suppose that $x \sim y$ and $y \sim z$.

By the definition of $\sim$, $5 \mid (3x + 7y)$ and $5 \mid (3y + 7z)$. So $3x + 7y = 5m$ and $3y + 7z = 5n$, for some integers m and n.

Adding these two equations together, we get $3x+7y+3y+7z = 5m+5n$. So $3x+10y+7z = 5(m+n)$. So $3x + 7z = 5(m + n - 2y)$.

$m + n - 2y$ is an integer, since m, n and y are integers. So this means that $5 \mid 3x + 7z$ and therefore $x \sim z$, which is what we needed to show.

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Let's define the relation $\succeq$ on $\mathbb{N}^2$ by

$(x, y) \succeq (a, b)$ if and only if $x - a \geq 2$ and $y \geq b$.

Prove that $\succeq$ is transitive.

Solution: Let (a, b), (x, y), and (c, d) be elements of X. Suppose that $(x, y) \succeq (a, b)$ and $(a, b) \succeq (c, d)$.

By the definition of $\succeq$, $(x, y) \succeq (a, b)$ implies that $x - a \geq 2$ and $y \geq b$. Similarly, $(a, b) \succeq (c, d)$ implies that $a - c \geq 2$ and $b \geq d$.

Since $y \geq b$ and $b \geq d$, $y \geq d$.

We know that $x - a \geq 2$ and $a - c \geq 2$, Adding these two equations, we get $(x - a) + (a - c) \geq 4$. So $x - c \geq 4$. So $x - c \geq 2$.

Therefore $x - c \geq 2$ and $y \geq d$. This implies that $(x, y) \succeq (c, d)$, which is what we needed to show.

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A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $r \leq q$ and $|c - d| + r \leq q$.

Prove that $\preceq$ is antisymmetric.

Solution: Let (c, r) and (d, q) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (c, r)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $r \leq q$ and $|c - d| + r \leq q$. Similarly, $(d, q) \preceq (c, r)$ means that $q \leq r$ and $|d - c| + q \leq r$.

Since $r \leq q$ and $q \leq r$, $q = r$. Substituting this into $|c - d| + r \leq q$, we get $|c - d| + r \leq r$. So $|c - d| \leq 0$. Since the absolute value of a real number cannot be negative, this means that $|c - d| = 0$, so $c = d$.

Since $q = r$ and $c = d$, $(c, r) = (d, q)$, which is what we needed to prove.

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Let $A = \mathbb{N} \times \mathbb{N}$, i.e. pairs of natural numbers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists an integer $n \geq 1$ such that $(x, y) = (np, nq)$.

Prove that $\gg$ is transitive.

Solution: Let (x, y), (p, q) and (a, b) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (a, b)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(a, b)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = ma$ and $q = mb$.

Combining these equations, we get $x = np = n(ma) = (nm)a$ and $y = nq = n(mb) = (nm)b$. Let $s = nm$. Since m and n are positive integers, so is s. But $(x, y) = (sa, sb)$. So $(x, y) \gg (a, b)$, which is what we needed to show.