CS 173, Fall 17 Examlet 5, Part A

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is onto. Recall that the max function returns the larger of its two inputs. Let's define $g : \mathbb{N}^2 \to \mathbb{N}$ by $g(x, y) = f(\max(x - 7, 0)) + f(y)$. Prove that g is onto.

   Solution: Let n be an arbitrary natural number.

   Since f is onto, there is a natural number p such that $f(p) = 0$. Similarly, there is also a natural number r such that $f(r) = n$.

   Then $g(p + 7, r) = f(\max(p, 0)) + f(r) = f(p) + f(r) = f(p) + f(r) = 0 + n = n$. So $(p + 7, r)$ is a pre-image for n, which is what we needed to find.

2. (5 points) Recall that $\mathbb{Z}^+$ is the set of positive integers. Let's define the function $h : (\mathbb{Z}^+)^2 \to \mathbb{R}$ such that $h(d, e) = 2^d + \frac{1}{e}$. Is h one-to-one? Briefly justify your answer.

   Solution: The image of h consists of a little sequence of numbers near each power of two (starting with 2). The offset (from the power of 2) starts off with 1 and then continues with a set of increasingly small positive fractions.

   Notice that consecutive powers of 2, i.e. 2^d as we vary d, are separated by at least 2. For a fixed first input d, the output values (as we vary the second input e) are all distinct and lie between 2^d and $2^d + 1$. So the values produced for one first input d are not only distinct from each other but well-separated from the values produced by other values of d. So h is one-to-one.

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1. (10 points) Let's define the function $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ by $f(x, y) = (x + y, 2x - 3y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of integers. Suppose that $f(x, y) = f(a, b)$.

   By the definition of f, $f(x, y) = f(a, b)$ implies that $(x + y, 2x - 3y) = (a + b, 2a - 3b)$. Therefore $x + y = a + b$ and $2x - 3y = 2a - 3b$.

   Subtracting twice $x + y = a + b$ from $2x - 3y = 2a - 3b$, we get $(2x - 3y) - (2x + 2y) = (2a - 3b) - (2a + 2b)$. Simplifying gives us $-5y = -5b$. So $y = b$.

   Substituting $y = b$ into $x + y = a + b$, we get $x + y = a + y$. So $x = a$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to prove.

2. (5 points) Suppose that $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ are functions. Let's define the function $f + g$ by $(f + g)(x) = f(x) + g(x)$. Adele claims that if f and g are onto, then $f + g$ is onto. Is this correct? Briefly explain why it is or give a counter-example.

   Solution: This is not correct. Suppose that $g(x) = f(x)$ for every input x. Then $(f + g)(x)$ is even for any input x. So the odd numbers aren't in the image of $f + g$ and therefore it isn't onto.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z} \to \mathbb{Z}^2$ by $g(n) = (|n|, f(n)|n|)$. Prove that g is one-to-one.

   Solution:

   Let p and q be integers. Suppose that $g(p) = g(q)$.

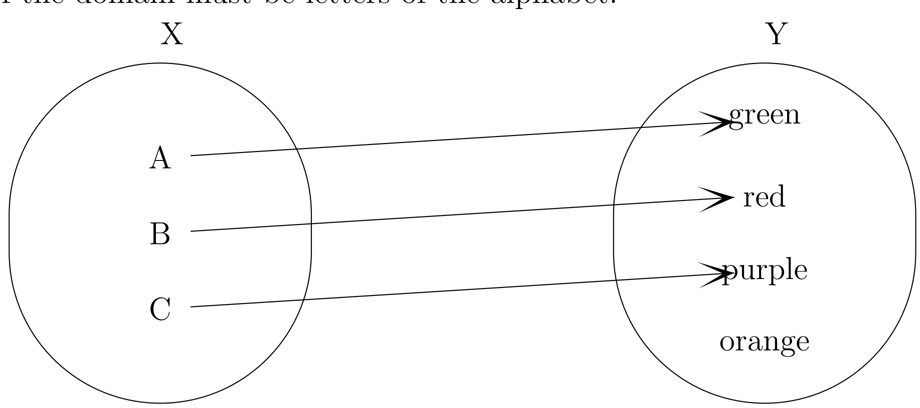
   By the definition of g, this means that $(|p|, f(p)|p|) = (|q|, f(q)|q|)$. So $|p| = |q|$ and $f(p)|p| = f(q)|q|$.

   Case 1: $|p| = 0$. Then $p = q = 0$. So $p = q$.

   Case 2: $|p|$ is non-zero. Substituting the first equation into the second, we get that $f(p)|p| = f(q)|p|$. So $f(p) = f(q)$. Since f is one-to-one, this means that $p = q$.

   So we've shown that $g(p) = g(q)$ implies that $p = q$, which means that g is one-to-one.

2. (5 points) Complete this picture to make an example of a function that is one-to-one but not onto, by adding elements to the domain and arrows showing how input values map to output values. The elements of the domain must be letters of the alphabet.



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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) If a is any real number, (a, ∞) is the set of all real numbers greater than a. Let's define the function $f : (0, \infty) \to (\frac{5}{4}, \infty)$ by $f(x) = \frac{5x^2+3}{4x^2}$. Prove that f is onto.

   Solution: Let $y \in (\frac{5}{4}, \infty)$. Consider $x = \sqrt{\frac{3}{4y-5}}$. Since $y > \frac{5}{4}$, $4y - 5$ is always positive. So x is well defined (no dividing by zero). And, also x is positive, so x comes from the domain of f.

   Then $x^2 = \frac{3}{4y-5}$.

   So $f(x) = \frac{5x^2+3}{4x^2} = \frac{5\frac{3}{4y-5}+3}{4\frac{3}{4y-5}}$

   Multiplying by $(4y - 5)$, we get $f(x) = \frac{5 \cdot 3+3(4y-5)}{4 \cdot 3} = \frac{15+12y-15}{12} = \frac{12y}{12} = y$

   Since y was chosen arbitrarily from the co-domain of f, we've shown that f is onto.

2. (5 points) What's wrong with this attempt to define $f \circ g$?

   > If $f : A \to B$ and $g : B \to C$ are functions, then $f \circ g$ is the function from A to C defined by $(f \circ g)(x) = f(g(x))$.

   Solution: This is applying the two functions in the wrong order. The domain of $f \circ g$ is stated to be A, so x must be an element of A. But we're applying g first and the inputs to g must come from the set B. So this is secretly assuming some overlap among the three sets, which you can't do in a general definition of composition.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x) + 2f(y) - 6$. Prove that g is onto.

   Solution: Let n be an arbitrary integer.

   Since f is onto, there is an integer input value y such that $f(y) = 3$. Similarly, there is an integer input value x such that $f(x) = n$.

   Now, consider (x, y). $g(x, y) = f(x) + 2f(y) - 6 = n + 2 \cdot 3 - 6 = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is increasing (but perhaps not strictly increasing). Dumbledore claims that f must be one-to-one. Is he correct? Briefly explain why he is or give a concrete counter-example.

   Solution: He's wrong. Suppose that $f(n) = 0$ for every input value n. Then f is (non-strictly) increasing, but not one-to-one.

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1. (10 points) Suppose that $h : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $f(x, y) = (h(x) - y, 3h(x) + 1)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(h(x) - y, 3h(x) + 1) = (h(p) - q, 3h(p) + 1)$. So $h(x) - y = h(p) - q$ and $3h(x) + 1 = 3h(p) + 1$.

   Since $3h(x) + 1 = 3h(p) + 1$, $3h(x) = 3h(p)$. So $h(x) = h(p)$. Since h is one-to-one, this means that $x = p$.

   We now know that $h(x) = h(p)$ and $h(x) - y = h(p) - q$. Combining these equations, we get that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Recall that $\mathbb{Z}^+$ is the set of positive integers. Let's define the function $h : (\mathbb{Z}^+)^2 \to \mathbb{R}$ such that $h(d, e) = 2^d + \frac{1}{e}$. Is h onto? Briefly justify your answer.

   Solution: The image of h consists of a little sequence of numbers near each power of two (starting wtih 2). The offset (from the power of 2) starts off with 1 and then continues with a set of increasingly small positive fractions. So every output value lies between a power of two and the next integer. This means that (for example) there are no output values between 17 (i.e. 16+1) and 32 (the next power of two). So h isn't onto.