CS 173, Fall 17 Examlet 7, Part A

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Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $2^{n+2} + 3^{2n+1}$ is divisible by 7, for all natural numbers n.

Solution:

Proof by induction on n.

Base case(s): At $n = 0$, $2^{n+2} + 3^{2n+1} = 2^2 + 3 = 7$ which is clearly divisible by 7.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $2^{n+2} + 3^{2n+1}$ is divisible by 7, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

At $n = k + 1$, $2^{n+2} + 3^{2n+1}$ is equal to $2^{k+3} + 3^{2k+3}$.

$2^{k+3} + 3^{2k+3} = 2 \cdot 2^{k+2} + 9 \cdot 3^{2k+1} = 2(2^{k+2} + 3^{2k+1}) + 7(3^{2k+1})$

By the inductive hypothesis, $2^{k+2} + 3^{2k+1}$ is divisible by 7. So $2(2^{k+2} + 3^{2k+1})$ is divisible by 7. $7(3^{3k+1})$ is divisible by 7 because it contains a literal factor of 7 and the rest of the expression (3^{3k+1}) is an integer. So the sum of these two terms must be divisible by 7.

Thus, $2^{k+3} + 3^{2k+3}$ is divisible by 7, which is what we needed to show.

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If f is a function, recall that f' is its derivative. Recall the product rule: if $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$. Assume we know that the derivative of $f(x) = x$ is $f'(x) = 1$.

Use (strong) induction to prove the following claim:

For any positive integer n, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Solution: Proof by induction on n.

Base case(s): $n = 1$. Then $f(x) = x$. So $f'(x) = 1$. But also $nx^{n-1} = 1 \cdot n^0 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$, for $n = 1, \ldots, k$.

Rest of the inductive step: Suppose that $f(x) = x^{k+1}$. Let $g(x) = x$ and $h(x) = x^k$. By the product rule $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Since $g(x) = x$, we know that $g'(x) = 1$. By the inductive hypothesis we know that $h'(x) = kx^{k-1}$.

So $f'(x) = g'(x)h(x) + g(x)h'(x) = 1 \cdot x^k + x \cdot kx^{k-1}$. Simplifying, we get $f'(x) = x^k + kx^k = (1+k)x^k$. So $f'(x) = (1 + k)x^k$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For any natural number n, $\sum_{p=0}^{n} 3(-1/2)^p = 2 + (-1/2)^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{p=0}^{n} 3(-1/2)^p = 3 \cdot (-1/2)^0 = 3$ and $2 + (-1/2)^n = 2 + (-1/2)^0 = 2 + 1 = 3$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=0}^{n} 3(-1/2)^p = 2 + (-1/2)^n$ for $n = 0, \ldots, k$.

Rest of the inductive step: From the inductive hypothesis, $\sum_{p=0}^{k} 3(-1/2)^p = 2 + (-1/2)^k$.

Then

$$\begin{aligned}
\sum_{p=0}^{k+1} 3(-1/2)^p &= (\sum_{p=0}^{k} 3(-1/2)^p) + 3(-1/2)^{k+1} \\
&= (2 + (-1/2)^k) + 3(-1/2)^{k+1} = 2 - 2(-1/2)^{k+1} + 3(-1/2)^{k+1} \\
&= 2 + (-1/2)^{k+1}
\end{aligned}$$

So $\sum_{p=0}^{k+1} 3(-1/2)^p = 2 + (-1/2)^{k+1}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all natural numbers n, $\sum_{p=0}^{n}(2p+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{p=1}^{n}(2p+1)^2 = 1^2 = 1$ and $\frac{(n+1)(2n+1)(2n+3)}{3} = \frac{1\cdot 1\cdot 3}{3} = 1$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
$\sum_{p=0}^{n}(2p+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for $n = 0, \ldots, k$.

Rest of the inductive step: From the inductive hypothesis, we know that
$\sum_{p=0}^{k}(2p+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.

Then

$$\begin{aligned}
\sum_{p=0}^{k+1}(2p+1)^2 &= \left(\sum_{p=0}^{k}(2p+1)^2\right) + (2(k+1)+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2 \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 = (2k+3)\frac{(k+1)(2k+1)+3(2k+3)}{3} \\
&= (2k+3)\frac{(2k^2+3k+1)+(6k+9)}{3} = (2k+3)\frac{2k^2+9k+10}{3} = \frac{(k+2)(2k+3)(2k+5)}{3}
\end{aligned}$$

So $\sum_{p=0}^{k+1}(2p+1)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$, which is what we needed to show

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Let's say that a set of polygonal regions in the plane is "properly colored" if regions sharing an edge never have the same color.

Suppose that we draw n lines in the plane, in general position (no lines are parallel, no point belongs to more than two lines). The lines divide up the plane into a set of regions. Use (strong) induction to prove that, for any positive integer n, this set of regions can be properly colored with two colors.

Solution: Proof by induction on n.

Base case(s): For $n = 1$, there is exactly one line dividing the plane. We can color one side of it red and the other side green.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the set of regions formed by n lines can be properly colored with two colors, for $n = 1, \ldots, k$.

Rest of the inductive step:

Suppose that we are given k+1 lines in general position. Pick an arbitrary line L and remove it. By the inductive hypothesis, we can find a coloring for the regions formed by the remaining lines in which adjacent regions always have different colors.

Now, add L back (keeping the regions colored). Swap the two colors on the regions to one side of L. Then

- Regions on the un-altered side of L have the colors they had before, so adjacent regions on this side have different colors.
- Regions on the altered side of L have exactly the opposite colors they had before, so adjacent regions on this side have different colors.
- Adjacent regions with L as their common boundary now have different colors, because one has its original color and the other has had its color swapped.

So we have a proper coloring of the regions formed by our set of $k + 1$ lines.

[OK if you used pictures to help explain the construction in the inductive step.]

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Use (strong) induction to prove the following claim:

Claim: For all integers a, b, n, $n \geq 1$, if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$.

Use this definition in your proof: $x \equiv y \pmod p$ if and only if $x = y + kp$ for some integer k.

Solution:

Proof by induction on n.

Base case(s): At $n = 1$, our claim becomes "if $a \equiv b \pmod 7$ then $a \equiv b \pmod 7$" which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for all integers a, b, n, where $n = 1, \ldots, k$,

a and b need to be introduced at some point in this proof, but there's several places you might do this. For example, you could say "let a and b be integers" right at the start. Then your inductive hypothesis would just be "if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for $n = 1, \ldots, k$." We won't get picky about this when grading.

Rest of the inductive step:

Let a and b be integers.

Suppose that $a \equiv b \pmod 7$. then $a = b + 7p$ for some integer p.

From the inductive hypothesis, we know that $a^k \equiv b^k \pmod 7$, So $a^k = b^k + 7q$ for some integer q.

Combining these two equations, we get that

$$a^{k+1} = (b + 7p)(b^k + 7q) = b^{k+1} + 7(pb^k + bq + 7pq)$$

$pb^k + bq + 7pq$ is an integer since p, q, and b are integers. So we know that $a^{k+1} \equiv b^{k+1} \pmod 7$, which is what we needed to prove.