CS 173, Fall 17 Examlet 8, Part A

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove that $a - b$ divides $a^n - b^n$, for any integers a and b and any natural number n.

Hint: $(a^n - b^n)(a + b) = (a^{n+1} - b^{n+1}) + ab(a^{n-1} - b^{n-1})$, for any real numbers a and b.

Solution: Let a and b be integers.

Proof by induction on n.

Base case(s):

At $n = 0$, $a^n - b^n = 1 - 1 = 0$, which is a multiple of any integer. So it's divisible by $a - b$.

At $n = 1$, $a^n - b^n = a - b$, so $a - b$ divides $a^n - b^n$.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $a - b$ divides $a^n - b^n$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

From the hint, we know that $a^{k+1} - b^{k+1} = (a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$

Notice that $(a^k - b^k)$ is divisible by $(a - b)$ by the inductive hypothesis. $(a + b)$ is an integer since a and b are integers. So $(a^k - b^k)(a + b)$ must be divisible by $(a - b)$.

Similarly, $(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$ by the inductive hypothesis. Also ab is an integer because a and b are integers. So $ab(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$.

So $(a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$ must be divisible by $(a - b)$, and therefore $a^{k+1} - b^{k+1}$ must be divisible by $(a - b)$, which is what we needed to show.

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(20 points) Suppose that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by

$f(n, 0) = f(n, n) = 1$, for any natural number n

$f(n, a) = f(n-1, a-1) + f(n-1, a)$, for all n and a such that $1 \le a \le n-1$

Use (strong) induction to prove that $f(n, a) = \frac{n!}{a!(n-a)!}$ for any natural numbers a and n, where $n \ge a$. Hint: use n as your induction variable. At each step, make sure the equations work for an arbitrary natural number $a \le n$.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, a must also be zero. So $f(n, a) = f(0, 0) = 1$. Also $\frac{n!}{a!(n-a)!} = \frac{0!}{0!0!} = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n, a) = \frac{n!}{a!(n-a)!}$ for $n = 0, \ldots, k-1$ and any natural number $a \le n$.

Rest of the inductive step: Let a be a natural number $\le k$. There are three cases:

Case 1: $a = k$. Then $f(k, a) = 1$ by the definition of f. Also $\frac{n!}{a!(n-a)!} = \frac{n!}{n!0!} = 1$. So the claim holds.

Case 2: $a = 0$. Then $f(k, a) = 1$ by the definition of f. Also $\frac{n!}{a!(n-a)!} = \frac{n!}{0!n!} = 1$. So the claim holds.

Case 3: $1 \le a \le k-1$. Then by the inductive hypothesis $f(k-1, a-1) = \frac{(k-1)!}{(a-1)!(k-a)!}$ and $f(k-1, a) = \frac{(k-1)!}{a!(k-1-a)!}$. Then

$$
\begin{aligned}
f(k, a) &= f(k-1, a-1) + f(k-1, a) = \frac{(k-1)!}{(a-1)!(k-a)!} + \frac{(k-1)!}{a!(k-1-a)!} \\
&= \frac{(k-1)!}{(a-1)!(k-a)!} + \frac{(k-1)!}{a!(k-1-a)!} = \frac{a(k-1)!}{a!(k-a)!} + \frac{(k-a)(k-1)!}{a!(k-a)!} \\
&= \frac{k(k-1)!}{a!(k-a)!} = \frac{k!}{a!(k-a)!}
\end{aligned}
$$

So $f(k, a) = \frac{k!}{a!(k-a)!}$, which is what we needed to show.

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(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2$ $\quad f(1) = 5$ $\quad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(0) = 2$ and $1 - 2^n + 2 \cdot 3^n = 1 - 1 + 2 = 2$

At $n = 1$, $f(1) = 5$ and $1 - 2^n + 2 \cdot 3^n = 1 - 2 + 6 = 5$

At $n = 2$, $f(2) = 15$ and $1 - 2^n + 2 \cdot 3^n = 1 - 4 + 18 = 15$

So the claim holds at all three values.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 1 - 2^n + 2 \cdot 3^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: By the definition of f and the inductive hypothesis, we get

$$\begin{aligned}
f(k) &= 6f(k-1) - 11f(k-2) + 6f(k-3) \\
&= 6(1 - 2^{k-1} + 2 \cdot 3^{k-1}) - 11(1 - 2^{k-2} + 2 \cdot 3^{k-2}) + 6(1 - 2^{k-3} + 2 \cdot 3^{k-3}) \\
&= (6 - 11 + 6) - (6 \cdot 2^{k-1} - 11 \cdot 2^{k-2} + 6 \cdot 2^{k-3}) + 2(6 \cdot 3^{k-1} - 11 \cdot 3^{k-2} + 6 \cdot 3^{k-3}) \\
&= 1 - (12 \cdot 2^{k-2} - 11 \cdot 2^{k-2} + 3 \cdot 2^{k-2}) + 2(18 \cdot 3^{k-2} - 11 \cdot 3^{k-2} + 2 \cdot 3^{k-2}) \\
&= 1 - 4 \cdot 2^{k-2} + 2 \cdot 9 \cdot 3^{k-2} = 1 - 2^k + 2 \cdot 2^k
\end{aligned}$$

So $f(k) = 1 - 2^k + 2 \cdot 2^k$, which is what we needed to show.

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(20 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is defined by

$f(0) = 0 \qquad f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$.

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Use (strong) induction to prove that $f(n) = \dfrac{a^n - b^n}{a-b}$.

First show that $a^2 = a + 1$ **and** $b^2 = b + 1$**: Solution:** Notice that $a^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1 + a$.

Similarly $b^2 = (\frac{1-\sqrt{5}}{2})^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = 1 + \frac{1-\sqrt{5}}{2} = 1 + b$.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(n) = 0$. Also $\frac{a^n - b^n}{a-b} = \frac{1-1}{a-b} = 0$.

At $n = 1$, $f(n) = 1$. Also $\frac{a^n - b^n}{a-b} = \frac{a-b}{a-b} = 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = \frac{a^n - b^n}{a-b}$, for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: In particular, by the inductive hypothesis, $f(k-1) = \frac{a^{k-1} - b^{k-1}}{a-b}$ and $f(k-2) = \frac{a^{k-2} - b^{k-2}}{a-b}$. Then

$$
\begin{aligned}
f(k) &= f(k-1) + f(k-2) = \frac{a^{k-1} - b^{k-1}}{a-b} + \frac{a^{k-2} - b^{k-2}}{a-b} \\
&= \frac{1}{a-b}(a^{k-1} - b^{k-1} + a^{k-2} - b^{k-2}) \\
&= \frac{1}{a-b}(a^{k-2}(a+1) - b^{k-2}(b+1)) \\
&= \frac{1}{a-b}(a^{k-2}(a^2) - b^{k-2}(b^2)) = \frac{1}{a-b}(a^k - b^k)
\end{aligned}
$$

So $f(k) = \frac{a^k - b^k}{a-b}$, which is what we needed to show.

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(20 points) Suppose that $g : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$g(1) = 1$

$g(2) = 8$

$g(n) = g(n-1) + 2g(n-2)$

Use (strong) induction to prove that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $g(n) = 1$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^0 + 2(-1) = 3 - 2 = 1$, so the claim holds.

At $n = 2$, $g(n) = 8$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n = 1, \ldots, k-1$.

Rest of the inductive step: In particular, by the inductive hypothesis, $g(k-1) = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$ and $g(k-2) = 3 \cdot 2^{k-3} + 2(-1)^{k-2}$.

So

$$
\begin{aligned}
g(k) &= g(k-1) - 2g(k-2) = (3 \cdot 2^{k-2} + 2(-1)^{k-1}) + 2(3 \cdot 2^{k-3} + 2(-1)^{k-2}) \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 6 \cdot 2^{k-3} + 4(-1)^{k-2} \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 3 \cdot 2^{k-2} - 4(-1)^{k-1} \\
&= 6 \cdot 2^{k-2} - 2(-1)^{k-1} = 3 \cdot 2^{k-1} + 2(-1)^k
\end{aligned}
$$

So $g(k) = 3 \cdot 2^{k-1} + 2(-1)^k$, which is what we needed to show.

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(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove that

$$\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m!}{n!(m-n)!}$$

for any positive integers m and n where $m \geq n$. Hint: use n as your induction variable. At each step, make sure the the equations work for an arbitrary integer $m \geq n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m+1-1}{1} = m$. And $\frac{m!}{n!(m-n)!} = \frac{m!}{1!(m-1)!} = m$. So the claim holds for any $m \geq n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\displaystyle\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m!}{n!(m-n)!}$ for $n = 1, \ldots, k$ and any integer $m \geq n$.

Rest of the inductive step: Let m be any integer $\geq n+1$. Notice that $m \geq n$. So, by the inductive hypothesis, $\displaystyle\prod_{p=1}^{k} \frac{m+1-p}{p} = \frac{m!}{k!(m-k)!}$. Then

$$\begin{aligned}
\prod_{p=1}^{k+1} \frac{m+1-p}{p} &= \frac{(m+1)-(k+1)}{k+1} \cdot \prod_{p=1}^{k} \frac{m+1-p}{p} = \frac{(m+1)-(k+1)}{k+1} \cdot \frac{m!}{k!(m-k)!} \\
&= \frac{m-k}{k+1} \cdot \frac{m!}{k!(m-k)!} = \frac{m!}{(k+1)!(m-k-1)!} \\
&= \frac{m!}{(k+1)!(m-(k+1))!}
\end{aligned}$$

So $\displaystyle\prod_{p=1}^{k+1} \frac{m+1-p}{p} = \frac{m!}{(k+1)!(m-(k+1))!}$, i.e. our claim holds at $n = k+1$, which is what we needed to prove.