# CS 173, Fall 17 Examlet 9, Part A

**Name:** ______________________________

**NetID:** ________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Here is a grammar $G$, with start symbol $S$ and terminal symbols $a$ and $b$.

$$S \rightarrow a\ S\ a \mid S\ a\ S \mid a\ N\ a$$
$$N \rightarrow a \mid b\ b$$

Use (strong) induction to prove that any tree of height $h$ matching (aka generated by) grammar $G$ has at least $h$ nodes with label $a$. Use $A(T)$ as shorthand for the number of a's in a tree $T$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

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(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Lemon tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Lemon tree of height $h$ contains at least $F_{h+1}$ nodes, where $F_k$ is the $k$th Fibonacci number. (Recall: $F_0 = 0$, $F_1 = F_2 = 1$)

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

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**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Suppose that $G$ is a connected graph. A Friendly coloring of $G$ labels each node orange or blue, such that

- If $G$ contains only one node, it is colored orange, and
- Otherwise, every node of $G$ is adjacent to at least one node of the opposite color.

Use (strong) induction to prove that any connected graph can be given a Friendly coloring. Hint: remove any node $x$ (no special properties required) and color the rest of the graph. What color pattern is required if $x$ is the only neighbor of another node?

The induction variable is named ______ and it is the ________________ of/in the graph.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

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(18 points) Recall that a node in a full binary tree must have either 0 or 2 children. A Happy tree is a full binary tree in which each node is colored orange or blue, such that:

- If $v$ is a leaf node, then $v$ may be colored orange or blue.
- If $v$ has two children of the same color, then $v$ is colored blue.
- If $v$ has two children of different colors, then $v$ is colored orange.

Use (strong) induction to show that the root of a Happy tree is blue if and only if the tree has an even number of orange leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

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(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Tidy tree is a full binary tree such that leaf nodes have label 0 and an internal node with label $x$ matches one of these patterns:

- The left subtree is a leaf and the right subtree has root label $y$, where $y \equiv x - 1 \pmod 3$, or
- The roots of both subtrees have label $y$, where $y \equiv x - 1 \pmod 3$.

Use (strong) induction to prove that the root label of every Tidy tree is congruent to $h \pmod 3$, where $h$ is the height of the tree. You may assume basic facts about congruence and modular arithmetic (e.g. congruence is transitive).

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

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**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Trim tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Trim tree of height $h$ contains at least $2^{h/2}$ leaves. You may use the fact that $\sqrt{2} > 1.4$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**