Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1+a_p) = 1+a_1$ and $1+\sum_{p=1}^{n} a_p = 1+a_1$ so $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any positive real numbers $a_1, \ldots, a_n$.

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be positive real numbers. By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1+a_p) \geq 1 + \sum_{p=1}^{k} a_p$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1+a_p) &= (1+a_{k+1})\prod_{p=1}^{k}(1+a_p) \\
&\geq (1+a_{k+1})(1+\sum_{p=1}^{k} a_p) = 1 + a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p + \sum_{p=1}^{k} a_p \\
&\geq 1 + a_{k+1} + \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 + \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1+a_p) \geq 1 + \sum_{p=1}^{k+1} a_p$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: For all integers $n \geq 2$, $(2n)! > 2^n n!$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $(2n)! = 4! = 24$. $2^n n! = 4 \cdot 2 = 8$. So $(2n)! > 2^n n!$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(2n)! > 2^n n!$ for all $n = 2, 3, \ldots, k$ for some integer $k \geq 2$.

Inductive Step: Notice that $2k + 1 \geq 1$ because k is positive. And $(2k)! > 2^k k!$ by the induction hypothesis.

So then

$(2(k+1))! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k)! > (2k+2)(2^k k!) = (k+1)2^{k+1}k! = 2^{k+1}(k+1)!$.

So $(2(k+1))! > 2^{k+1}(k+1)!$ which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any sets $A_1, A_2, \ldots, A_n$, $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ the claim reduces to $|A_1| \leq |A_1|$, which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$, for any sets $A_1, A_2, \ldots, A_n$, where $n = 1, 2, \ldots, k$.

Inductive Step: Let $A_1, A_2, \ldots, A_{k+1}$ be sets. Let $S = A_1 \cup A_2 \cup \ldots \cup A_k$.

We know that $|S \cup A_{k+1}| = |S| + |A_{k+1}| - |S \cap A_{k+1}|$ by the Inclusion-Exclusion formula. So $|S \cup A_{k+1}| \leq |S| + |A_{k+1}|$ because $|S \cap A_{k+1}|$ cannot be negative.

By the inductive hypothesis $|S| = |A_1 \cup A_2 \cup \ldots \cup A_k| \leq |A_1| + |A_2| + \ldots + |A_k|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| = |S \cup A_{k+1}| \leq |S| + |A_{k+1}| \leq (|A_1| + |A_2| + \ldots + |A_k|) + |A_{k+1}|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| \leq |A_1| + |A_2| + \ldots + |A_{k+1}|$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim. You may use the fact that $\sqrt{2} \leq 1.5$.

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq 2\sqrt{n+1} - 2$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$. Also $2\sqrt{n+1} - 2 = 2\sqrt{2} - 2 \leq 2 \cdot 1.5 - 2 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq 2\sqrt{n+1} - 2$ for $n = 1, 2, \ldots, k$.

Inductive Step: First, notice that $(\sqrt{k+1} - \sqrt{k+2}\,)^2 \geq 0$. Multiplying this out gives us $(k+1) - 2\sqrt{k+1}\sqrt{k+2} + (k+2) \geq 0$. So $2k + 3 \geq 2\sqrt{k+1}\sqrt{k+2}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq 2\sqrt{k+1} - 2$. So then

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} &= \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \frac{1}{\sqrt{k+1}} + 2\sqrt{k+1} - 2 \\
&= \frac{1}{\sqrt{k+1}} + \frac{2(k+1)}{\sqrt{k+1}} - 2 = \frac{1 + 2(k+1)}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2 \\
&\geq \frac{2\sqrt{k+1}\sqrt{k+2}}{\sqrt{k+1}} - 2 = 2\sqrt{k+2} - 2
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \geq 2\sqrt{k+2} - 2$, which is what we needed to show.

CS 173, Fall 17 Examlet 10, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{N}$ be defined by

$f(1) = 0$

$f(n) = 1 + f(\lfloor n/2 \rfloor)$, for $n \geq 2$,

Use (strong) induction on n to prove that $f(n) \leq \log_2 n$ for any positive integer n. You cannot assume that n is a power of 2. However, you can assume that the log function is increasing (if $x \leq y$ then $\log x \leq \log y$) and that $\lfloor x \rfloor \leq x$.

Solution:

Proof by induction on n.

Base Case(s):

$f(1) = 0$ and $\log_2 1 = 0$ So $f(1) \leq \log_2 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \leq \log_2 n$ for $n = 1, \ldots, k-1$.

Inductive Step:

We can assume that $k \geq 2$ (since we did $n = 1$ for the base case). So $\lfloor k/2 \rfloor$ must be at least 1 and less than k. Therefore, by the inductive hypothesis, $f(\lfloor k/2 \rfloor) \leq \log_2(\lfloor k/2 \rfloor)$.

We know that $f(k) = 1 + f(\lfloor k/2 \rfloor)$, by the definition of f. Substituting the result of the previous paragraph, we get that $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$.

$\lfloor k/2 \rfloor \leq k/2$. So $\log_2(\lfloor k/2 \rfloor) \leq \log_2(k/2) = (\log_2 k) + (\log_2 1/2) = (\log_2 k) - 1$.

Since $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$ and $\log_2(\lfloor k/2 \rfloor) \leq (\log_2 k) - 1$, $f(k) \leq 1 + (\log_2 k) - 1 = (\log_2 k)$. This is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n, and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1-a_p) \geq 1 - \sum_{p=1}^{n} a_p$$

Solution:

This problem should have also required that $a_1, \ldots, a_n$ be ≤ 1. This shouldn't have a major impact on grading because it looks like many folks assumed the critical step would work.

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1-a_p) = 1-a_1$ and $1-\sum_{p=1}^{n} a_p = 1-a_1$ so $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1-a_p) \geq 1 - \sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any real numbers $a_1, \ldots, a_n$ between 0 and 1 (inclusive).

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be real numbers between 0 and 1 (inclusive). By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1-a_p) \geq 1 - \sum_{p=1}^{k} a_p$. Since $(1-a_{k+1})$ is positive, this means that $(1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p)$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1-a_p) &= (1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \\
&\geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p) \;=\; 1 - a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p - \sum_{p=1}^{k} a_p \\
&\geq 1 - a_{k+1} - \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 - \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1-a_p) \geq 1 - \sum_{p=1}^{k+1} a_p$, which is what we needed to show.