# CS 173, Fall 17 Examlet 10, Part B

**Name:**____________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (9 points) Fill in key facts about the recursion tree for $T$, assuming that $n$ is a power of 3.

   $T(3) = 7 \qquad T(n) = 4T\left(\frac{n}{3}\right) + 5n$

   (a) The height: $\log_3 n - 1$

   (b) Value in each node at level $k$: Each node at level $k$ contains the value $\frac{5n}{3^k}$.

   (c) Sum of the work in all the leaves (please simplify): The number of leaves is $4^{\log_3 n - 1} = \frac{1}{4}4^{\log_3 n}$
   $4^{\log_3 n} = 4^{\log_4 n \log_3 4} = (4^{\log_4 n})^{\log_3 4} == n^{\log_3 4}$
   So the work at the leaves is $\frac{7}{4}n^{\log_3 4}$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that $f$ is to the left of $g$ if and only if $f(n) \ll g(n)$.

   $(3^n)^2 \quad 10 \quad 0.001n^3 \quad 30\log n \quad n\log(n^7) \quad 8n! + 18 \quad 3n^2$

   **Solution:**

   $10 \ll 30\log n \ll n\log(n^7) \ll 3n^2 \ll 0.001n^3 \ll (3^n)^2 \ll 8n! + 18$

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that $f$, $g$, and $h$ are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x) \ll f(x))$. Must $f(x) + g(x)$ be $O(h(x))$?

   **Solution:** This is true. Since $g(x)$ is asymptotically smaller than $f(x)$, $f(x) + g(x)$ grows at the same rate as $f(x)$. We know this is $O(h(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $T(1) = d$<br>$T(n) = 3T(n/3) + c$ | $\Theta(\log n)$ ☐<br>$\Theta(n^2)$ ☐ | $\Theta(\sqrt{n})$ ☐<br>$\Theta(n^3)$ ☐ | $\Theta(n)$ $\surd$<br>$\Theta(2^n)$ ☐ | $\Theta(n \log n)$ ☐<br>$\Theta(3^n)$ ☐ |
| $T(1) = c$<br>$T(n) = 3T(n/3) + n$ | $\Theta(\log n)$ ☐<br>$\Theta(n^2)$ ☐ | $\Theta(\sqrt{n})$ ☐<br>$\Theta(n^3)$ ☐ | $\Theta(n)$ ☐<br>$\Theta(2^n)$ ☐ | $\Theta(n \log n)$ $\surd$<br>$\Theta(3^n)$ ☐ |
| $n!$ | $O(2^n)$ ☐ | $\Theta(2^n)$ ☐ | neither of these $\surd$ | |
| $n^{log_2 4}$ grows | faster than $n^2$ ☐<br>at the same rate as $n^2$ $\surd$ | slower than $n^2$ ☐ | | |

# CS 173, Fall 17 Examlet 10, Part B

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that $f$, $g$, and $h$ are functions from the reals to the reals, such that $f(x)$ is $\Theta(h(x))$, $g(x)$ is $\Theta(h(x))$, and $f(x) > g(x)$ for any input $x$. Must $f(x) - g(x)$ be $\Theta(h(x))$?

   **Solution:** This is false.

   Suppose that $g(x) = h(x) = x^2$ and $f(x) = x^2 + x$. Then $f(x) - g(x) = x$, which is not $\Theta(x^2)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 2T(n/2) + n$

| | | | |
|---|---|---|---|
| $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☐ | $\Theta(n \log n)$ ☑ |
| $\Theta(n^2)$ ☐ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☐ | $\Theta(3^n)$ ☐ |

$T(1) = d$
$T(n) = 2T(n-1) + c$

| | | | |
|---|---|---|---|
| $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☐ | $\Theta(n \log n)$ ☐ |
| $\Theta(n^2)$ ☐ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☑ | $\Theta(3^n)$ ☐ |

$n^{log_2 3}$ grows

faster than $n$ ☑ slower than $n$ ☐
at the same rate as $n$ ☐

Suppose $f(n)$ is $\Theta(g(n))$.
Will $g(n)$ be $\Theta(f(n))$?

no ☐ perhaps ☐ yes ☑

# CS 173, Fall 17 Examlet 10, Part B

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) You found the following claim on a hallway whiteboard. Suppose that $f$ and $g$ are increasing functions from the reals to the reals, for which all output values are $> 1$. If $f(x)$ is $O(g(x))$, then $\log(f(x))$ is $O(\log(g(x)))$. Is this true? Briefly justify your answer.

   **Solution:**

   Yes, it is true. Suppose that $f(x)$ is $O(g(x))$. Then there are positive reals $c$ and $k$ such that $f(x) \leq cg(x)$ for all $x \geq k$. Then $\log(f(x)) \leq \log c + \log(g(x))$ for all $x \geq k$. Since $g(x)$ is an increasing function and $\log c$ isn't, There is some $K \geq k$ such that $\log c \leq \log(g(x))$. So then $\log(f(x)) \leq 2\log(g(x))$ for all $x \geq K$. So $\log(f(x))$ is $O(\log(g(x)))$.

   [You don't need this much technical detail for full credit. We can ignore the other inequality from the definition of big-O because the two functions always output positive values.]

2. (8 points) Check the (single) box that best characterizes each item.

| | | | |
|---|---|---|---|
| $3^n$ is | $\Theta(2^n)$ ☐ | $O(2^n)$ ☐ | neither of these ☑ |

| | | |
|---|---|---|
| Dividing a problem of size $n$ into $m$ subproblems, each of size $n/k$, has the best big-$\Theta$ running time when | $k < m$ ☐ | $k = m$ ☐ |
| | $k > m$ ☑ | $km = 1$ ☐ |

| | | | | |
|---|---|---|---|---|
| $T(1) = d$ <br> $T(n) = T(n/2) + n$ | $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☑ | $\Theta(n \log n)$ ☐ |
| | $\Theta(n^2)$ ☐ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☐ | $\Theta(3^n)$ ☐ |

| | | | | |
|---|---|---|---|---|
| $T(1) = d$ <br> $T(n) = T(n-1) + n$ | $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☐ | $\Theta(n \log n)$ ☐ |
| | $\Theta(n^2)$ ☑ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☐ | $\Theta(3^n)$ ☐ |

**Name:**__________________________________________

**NetID:**__________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for $T$, assuming that $n$ is a power of 2.

   $T(4) = 7 \qquad T(n) = 5T\left(\frac{n}{2}\right) + n$

   (a) The height: $\log_2 n - 2$

   (b) The number of leaves (please simplify): $5^{\log_2 n - 2} = \frac{1}{25}5^{log_2 n} = \frac{1}{25}5^{log_5 n \log_2 5} = \frac{1}{25}n^{\log_2 5}$

   (c) Value in each node at level k: $\frac{n}{2^k}$

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that $f$ is to the left of $g$ if and only if $f(n) \ll g(n)$.

   $30\log(n^{17}) \qquad \sqrt{n} + n! + 18 \qquad \frac{n \log n}{7} \qquad (10^{10^{10}})n \qquad 0.001n^3 \qquad 2^n \qquad 8n^2$

   **Solution:**

   $30\log(n^{17}) \ll (10^{10^{10}})n \ll \frac{n \log n}{7} \ll 8n^2 \ll 0.001n^3 \ll 2^n \ll \sqrt{n} + n! + 18$

# CS 173, Fall 17 Examlet 10, Part B

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that $f$ and $g$ are functions from the reals to the reals, such that $f$ is $\Theta(g)$. Must $g$ be $O(f)$?

   **Solution:** This is true. The definition of $\Theta$ is that the big-O relationship holds in both directions.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 2T(n/2) + n^2$

| | | | |
|---|---|---|---|
| $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☐ | $\Theta(n \log n)$ ☐ |
| $\Theta(n^2)$ ☑ √ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☐ | $\Theta(3^n)$ ☐ |

$T(1) = d$
$T(n) = T(n-1) + c$

| | | | |
|---|---|---|---|
| $\Theta(\log n)$ ☐ | $\Theta(\sqrt{n})$ ☐ | $\Theta(n)$ ☑ √ | $\Theta(n \log n)$ ☐ |
| $\Theta(n^2)$ ☐ | $\Theta(n^3)$ ☐ | $\Theta(2^n)$ ☐ | $\Theta(3^n)$ ☐ |

$n^{log_4 2}$ grows

| | |
|---|---|
| faster than $n^2$ ☐ | slower than $n^2$ ☑ √ |
| at the same rate as $n^2$ ☐ | |

$\log_5 n$ is

| | | |
|---|---|---|
| $\Theta(\log_3 n)$ ☑ √ | $O(\log_3 n)$ ☐ | neither of these ☐ |