CS 173, Fall 17 Examlet 12, Part A

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(8 points) Marsha has 25 cups, identical except that 10 are red, 8 are blue, and 7 are green. How many ways can she make an (ordered) sequence of 25 cups?

Solution: In the sequence, she has $\binom{25}{10}$ choices for where to put the red cups. Then she has $\binom{15}{8}$ choices for where to put the blue cups. So the total number of options is $\binom{25}{10}\binom{15}{8}$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a dorm room d, such that d has green walls and d has no window.

Solution: For every dorm room d, d has walls that aren't green or d has a window.

(2 points) Check the (single) box that best characterizes each item.

Suppose you want to take 5 classes next semester, out of 25 classes being offered. You must take STATS 100. How many different choices do you have?

$\frac{24!}{20!}$	☐	$\frac{24!}{20!4!}$	☑ √
$\frac{(24+3)!}{24!3!}$	☐	$\frac{25!}{20!5!}$	☐

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(9 points) Use proof by contradiction to show that, for any graph G and any two nodes a and b, the shortest walk from a to b does not contain any repeated nodes.

Solution: Suppose not. That is, suppose we have a graph G, two nodes a and b, and the shortest walk W from a to b contains at least one repeated node.

Since W contains a repeated pair of nodes, it must look like $a = w_1, w_2, \ldots, w_{n-1}, w_n = b$. Suppose that $w_i = w_k$, where $i < k$. We can then make a new walk W' from a to b by removing the nodes between w_i and w_k, merging w_i with w_k. W' is shorter than W, contradicting our assumption that W was the shortest walk between these two nodes.

(6 points) Suppose a set S has 11 elements. How many subsets of S have an even number of elements? Express your answer as a summation. Briefly justify or show work.

Solution: Subsets of S having an even number of elements would have $0, 2, 4, 6, 8$ or 10 elements. There are $\binom{11}{0}$ subset of S with no elements, $\binom{11}{2}$ subsets of S with 2 elements, and so on. So the number of subsets of S having an even number of elements is

$$\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \binom{11}{6} + \binom{11}{8} + \binom{11}{10} = \sum_{i=0}^{5} \binom{11}{2i}.$$

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(8 points) Suppose we know (e.g. from the binomial theorem) that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Use this to show that $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$. Show your work.

Solution:

$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

So $\sum_{k=1}^{n} k \cdot \binom{n}{k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} = n \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} = n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1}$

Changing the index of the summation and using the given identity gives us

$n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} = n \cdot 2^{n-1}$

Combining this with the previous equation, we get $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every tree t, if t grows in Canada, then t is not tall or t is a conifer.

Solution: There is a tree t, such that g is tall and t is not a conifer, bug t grows in Canada.

(2 points) Check the (single) box that best characterizes each item.

How many ways can I choose a set of 5 bagels, if there are 10 types of bagels and all 5 bagels must be different types?

$\frac{10!}{5!5!}$ ☑ √	$\frac{14!}{10!4!}$ ☐	$\frac{14!}{9!5!}$ ☐
$\frac{15!}{10!5!}$ ☐	10^5 ☐	5^{10} ☐

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(9 points) Researchers recorded phone calls between pairs of two (different) people, never repeating the same pair of people. The calls used n people ($n \geq 2$), but it's possible some people were not in any conversation. Use proof by contradiction to show that two people were in the same number of conversations.

Solution: Suppose not. That is, suppose that each of the n people was in a different number of conversations. For each person, the minimum number of conversations is zero and the maximum is $n-1$. Since there are exactly n numbers between 0 and $n-1$, there's some person P who wasn't in any conversation and another person Q who was in $n-1$ conversations. But this is a contradiction. If Q talked to $n-1$ people, then Q must have talked to P, contradicting the fact that P didn't talk to anyone.

(6 points) Use the binomial theorem to find a closed form for the summation $\sum_{k=0}^{n} \binom{n}{k}$. Make sure it's clear how you used the theorem.

Solution: The binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$.

Setting $x = y = 1$ gives us $2^n = (1+1)^n = \sum_{k=0}^{n} (1)^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}$

That is $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

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(9 points) Use proof by contradiction to show that, for any integer n, at least one of the three integers n, $2n+1$, $4n+3$ is not divisible by 7.

Solution: Suppose not. That is, suppose that n, $2n+1$, $4n+3$ are all divisible by 7. Then their sum $n+(2n+1)+(4n+3)$ must be divisible by 7. So $7n+4$ must be divisible by 7. But then 4 would need to be divisible by 7, which isn't true.

Since its negation led to a contradiction, our original claim must have been true.

(6 points) Suppose a set S has 11 elements. How many subsets of S have five or fewer elements? Express your answer as a summation. Briefly justify or show work.

Solution: There are $\binom{11}{0}$ subset of S with no elements, $\binom{11}{2}$ subsets of S with 2 elements, and so on. So the number of subsets of S having five or fewer elements is

$$\binom{11}{0}+\binom{11}{1}+\binom{11}{2}+\binom{11}{3}+\binom{11}{4}+\binom{11}{5}=\sum_{i=0}^{5}\binom{11}{i}.$$

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(8 points) Recall that nodes in a full binary tree have either zero or two children. Suppose we are building a full binary tree with unlabelled nodes whose leaves are all at levels k or $k+1$, with p leaves at level $k+1$. How many different ways can we construct such a tree? Briefly justify your answer.

Solution: Notice that nodes in each level come in pairs, because the tree is full. So p must be even. Let $p = 2n$.

The only thing we get to choose when constructing these trees is which nodes at level k have two children (vs. zero children). There are 2^k nodes at level k, of which n will have children. So we have $\binom{2^k}{n}$ options for constructing the tree.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

If it is raining, then there is a cyclist c such that c is getting wet.

Solution: It is raining and for every cyclist c, c is not getting wet.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (one or more of each variety).

$\binom{16}{3}$ ☑	$\binom{16}{4}$ ☐	$\binom{20}{3}$ ☐
$\binom{20}{4}$ ☐	$\binom{21}{3}$ ☐	4^{17} ☐