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(15 points) Notice that, for any integer p, $\lfloor p \rfloor = \lfloor p + \frac{1}{2} \rfloor = p$. Using this fact and your best mathematical style, prove the following claim:

For any integer n, if n is odd, then $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$

Solution: Let n be an integer and suppose that n is odd. Since n is odd, we can write $n = 2k + 1$, where k is an integer.

Looking at the left side of our equation, we have $\lfloor \frac{n}{2} \rfloor^2 + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor^2 + \lfloor \frac{2k+1}{2} \rfloor = \lfloor k + \frac{1}{2} \rfloor^2 + \lfloor k + \frac{1}{2} \rfloor = k^2 + k$

On the right side, we have $\frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{(2k+1)^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{4k^2+4k+1}{2} \right\rfloor = \frac{1}{2} \lfloor 2k^2 + 2k + \frac{1}{2} \rfloor = \frac{1}{2}(2k^2 + 2k) = k^2 + k$. (Noting that $2k^2 + 2k$ must be an integer because k is an integer.)

So $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$ and therefore $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

> Claim: for all integers a, b, c, d, j, and k (j and k positive), if $a \equiv b \pmod{j}$, $c \equiv d \pmod{k}$, and $j \mid k$, then $a + c \equiv b + d \pmod{j}$.

Solution: Let a, b, c, d, j, and k (j and k positive). Suppose that $a \equiv b \pmod{j}$, $c \equiv d \pmod{k}$, and $j \mid k$.

Using the given definition of congruence, since $a \equiv b \pmod{j}$, $a = b + nj$ for some integer n. Similarly, since $c \equiv d \pmod{k}$, $c = d + mk$ for some integer m.

Adding these two equations together, we get $a + c = (b + nj) + (d + mk)$.

Since $j \mid k$, $k = pj$ by the definition of divides. Substituting this into the above equation, we get:

$a + c = (b + nj) + (d + mpj) = b + d + (n + mp)j$

$n + mp$ is an integer because n, m, and p are all integers. So by the definition of congruence, $a + c \equiv b + d \pmod{j}$.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all rational numbers x, y and z, if y is non-zero, then $5(\frac{x}{y}) - 2z$ is rational.

Solution: Let x, y and z be rational numbers and suppose that y is non-zero.

By the definition of rational, $x = \frac{a}{b}$, $y = \frac{c}{d}$ and $z = \frac{e}{f}$, where the numbers a to f are all integers and b, d, and f are non-zero. Since y is non-zero, c must also be non-zero.

We can then compute

$$\begin{aligned} 5(\frac{x}{y}) - 2z &= 5(\frac{\frac{a}{b}}{\frac{c}{d}}) - 2\frac{e}{f} \\ &= 5(\frac{ad}{bc}) - 2\frac{e}{f} \\ &= \frac{5adf - 2ebc}{bcf} \end{aligned}$$

Let $p = 5adf - 2ebc$ and $q = bcf$. p and q are integers because a, b, c, d, e, and f are all integers. Futhermore, q is non-zero, because b, c, and f are all non-zero.

Therefore, $5(\frac{x}{y}) - 2z = \frac{p}{q}$, where p and q are integers and q is non-zero. So $5(\frac{x}{y}) - 2z$ is rational.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim by contrapositive.

> For all real numbers x and y, if x is not rational, then $2x + 3y$ is not rational or y is not rational.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if $2x+3y$ is rational and y is rational, then x is rational.

Let x and y be real numbers. Suppose that $2x + 3y$ is rational and y is rational. Then $2x + 3y = \frac{a}{b}$ and $y = \frac{m}{n}$, where a, b, m, n are integers, b and n non-zero.

Then $2x + 3\frac{m}{n} = \frac{a}{b}$

So $2x = \frac{a}{b} - \frac{3m}{n} = \frac{an-3bm}{bn}$

So $x = \frac{an-3bm}{2bn}$

$an - 3bm$ and $2bn$ are both integers because a, b, m, n are integers. Also $2bn$ is non-zero because b and n are non-zero. So x is rational.

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(15 points) Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases.

For any integer n, $n^2 + 2$ is not divisible by 4.

Solution: Let n be an integer. From the Division Algorithm (aka definition of remainder), we know that there are integers q and r such that $n = 4q + r$.

There are four cases, depending on what the remainder r is:

Case 1: $n = 4q$. Then $n^2 + 2 = 16q^2 + 2 = 4(4q^2) + 2$.

Case 2: $n = 4q + 1$. Then $n^2 + 2 = 16q^2 + 8q + 3 = 4(4q^2 + 2q) + 3$.

Case 3: $n = 4q + 2$. Then $n^2 + 2 = 16q^2 + 16q + 6 = 4(4q^2 + 4q + 1) + 2$.

Case 4: $n = 4q + 3$. Then $n^2 + 2 = 16q^2 + 24q + 11 = 4(4q^2 + 6q + 2) + 3$.

In all four cases, the remainder of n divided by 4 is not zero, so n isn't divisible by 4.

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(15 points) A triple (a, b, c) of positive integers is Pythagorean if $a^2 + b^2 = c^2$. Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For any Pythagorean triple (a, b, c), if c is odd, then a is even or b is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for any Pythagorean triple (a, b, c), if a and b are odd, then c is even.

So suppose (a, b, c) is Pythagorean and a and b are odd. Then $a^2 + b^2 = c^2$ by the definition of Pythagorean. Also, by the definition of odd, $a = 2m + 1$ and $b = 2p + 1$, where m and n are integers.

Then $c^2 = a^2 + b^2 = (2m+1)^2 + (2p+1)^2 = (4m^2+4m+1) + (4p^2+4p+1) = 2(2m^2+2m+2p^2+2p+1)$

Let $t = 2m^2 + 2m + 2p^2 + 2p + 1$. t is an integer because m and p are integers. And $c^2 = 2t$. So c^2 is even.

[We hadn't actually intended you to have to prove the additional part shown below, so the TAs have instructions to be nice about the grading.]

If c was odd, then we'd have $c = 2k + 1$. This would mean that $c^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ which is odd. So c has to be even.