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Suppose that T is a relation on the integers which is transitive. Let's define a relation R on pairs of integers such that $(p,q)R(a,b)$ if and only if $(a+b)T(p+q)$. Prove that R is transitive.

Solution: Let (a,b), (p,q), and (m,n) be pairs of integers. Suppose that $(a,b)R(p,q)$ and $(p,q)R(m,n)$.

By the definition of R, this means that $(a,b)R(p,q)$ means that $(p+q)T(a+b)$ Similarly,$(p,q)R(m,n)$ means that $(m+n)T(p+q)$.

Because T is transitive, $(m+n)T(p+q)$ and $(p+q)T(a+b)$ implies that $(m+n)T(a+b)$.

By the definition of R, $(m+n)T(a+b)$ implies that $(a,b)R(m,n)$, which is what we needed to show.

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Let T be the relation defined on $\mathbb{Z}^2$ by

$$(x, y)T(p, q) \text{ if and only if } x < p \text{ or } (x = p \text{ and } y \leq q)$$

Prove that T is antisymmetric.

Solution:

Let (x, y) and (p, q) be pairs of integers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$. By the definition of T $(x, y)T(p, q)$ means that $x < p$ or $(x = p$ and $y \leq q)$. Similarly, $(p, q)T(x, y)$ means that $p < x$ or $(p = x$ and $q \leq y)$.

There are four cases:

Case 1: $x < p$ and $p < x$. This is impossible.

Case 2: $x < p$ and $p = x$ and $q \leq y$. Also impossible.

Case 3: $p < x$ and $x = p$ and $y \leq q$. Impossible as well.

Case 4: $x = p$ and $y \leq q$ and $p = x$ and $q \leq y$. Since $y \leq q$ and $q \leq y$, $x = y$. So we have $(x, y) = (p, q)$.

$(x, y) = (p, q)$ is true, which is what we needed to show.

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Let $A = \{(a, b) \in \mathbb{R}^2 \ : \ a > 0, b > 0, \text{ and } a + b = 90\}$. That is, an element of A is a pair of positive reals that sum to 90 (e.g. the non-right angles in a right triangle). Now, let's define a relation $\sim$ on A as follows:

$(a, b) \sim (p, q)$ if and only if $a = p$ or $a = q$.

Prove that $\sim$ is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b) \sim (p, q)$ and $(p, q) \sim (m, n)$.

Since $(a, b) \sim (p, q)$, $a = p$ or $a = q$. Since $(p, q) \sim (m, n)$, $p = m$ or $p = n$.

First, notice that $q = 90 - p$, $n = 90 - m$, and $m = 90 - n$.

There are four cases.

Case 1: $a = p$ and $p = m$. Then $a = m$

Case 2: $a = p$ and $p = n$. Then $a = n$.

Case 3: $a = q$ and $p = m$. Then $a = 90 - p = 90 - m = n$. So $a = n$.

Case 4: $a = q$ and $p = n$. Then $a = 90 - p = 90 - n = m$. So $a = m$.

In all four cases, $a = m$ or $a = n$. So, by the definition of $\sim$, we have $(a, b) \sim (m, n)$, which is what we needed to prove.

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For any two real numbers with $a \leq b$, the closed interval $[a, b]$ is defined by $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$. Let J be the set containing all closed intervals $[a, b]$. Let's define the relation F on J as follows:

$[s, t]F[p, q]$ if and only if $q \leq s$

Prove that F is antisymmetric.

Solution: Let $[s, t]$ and $[p, q]$ be two closed intervals. Suppose that $[s, t]F[p, q]$ and $[p, q]F[s, t]$.

By the definition of F, this means that $q \leq s$ and $t \leq p$. By the definition of closed interval, $s \leq t$ and $p \leq q$. So we have

$$p \leq q \leq s \leq t \leq p$$

So $p = q = s = t$ and therefore $[s, t] = [p, q]$, which is what we needed to show.

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $x \leq p$ and $xy \leq pq$

Prove that T is antisymmetric.

Solution: Let (x, y) and (p, q) be elements of A. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$.

By the definition of T, $(x, y)T(p, q)$ implies that $x \leq p$ and $xy \leq pq$.

Similarly $(p, q)T(x, y)$ implies that that $p \leq x$ and $pq \leq xy$.

Since $x \leq p$ and $p \leq x$, $x = p$. Since $xy \leq pq$ and $pq \leq xy$, $xy = pq$.

Notice that x and o are positive, by the definition of A. So $x = p$ and $xy = pq$ implies that $y = q$.

We now know that $x = p$ and $y = q$. So therefore $(x, y) = (p, q)$, which is what we needed to show.

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A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval "starts" relation $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $c + q = d + r$ and $c + r \leq d + q$

Prove that $\preceq$ is transitive.

Solution: Let (c, r), (d, q), and (f, s) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (f, s)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $c+q = d+r$ and $c+r \leq d+q$ Similarly, $(d, q) \preceq (f, s)$ means that $d + s = f + q$ and $d + q \leq f + s$.

Since $c + r \leq d + q$ and $d + q \leq f + s$, $c + r \leq f + s$.

We also know that $c+q = d+r$ and $d+s = f+q$. We can rewrite the second equation as $d = f+q-s$. Substituting this into the first equation, we get $c+q = (f+q-s)+r$. So $c = f-s+r$. So $c+s = f+r$.

Since $c + s = f + r$ and $c + r \leq f + s$, $(c, r) \preceq (f, s)$, which is what we needed to show.