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NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = (1 - f(x))f(y)$. Prove that g is onto.

   Solution: Suppose that n is a natural number.

   Since f is onto, there is a natural number p such that $f(p) = 0$. Then $(1 - f(p)) = 1$

   Also since f is onto, there is a natural number q such that $f(q) = n$.

   Now consider the pair (p, q). $g(p, q) = (1 - f(p))f(q) = 1 \cdot n = n$. So (p, q) is a pre-image for n, which is what we needed to find.

2. (5 points) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is one-to-one but not onto. Be specific.

   Solution: Let $f(n) = n + 1$ if $n \geq 0$, and $f(n) = n - 1$ if $n < 0$. Then f is one-to-one. However, it's not onto because 0 isn't in the image of f.

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1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{x^2+2}{3x^2}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{x^2+2}{3x^2} = \frac{y^2+2}{3y^2}$.

   Multiplying both sides by the denominators, we get $(x^2+2)(3y^2) = (3x^2)(y^2+2)$. That is, $3x^2y^2 + 6y^2 = 3x^2y^2 + 6x^2$. Thus $6y^2 = 6x^2$. So $y^2 = x^2$. Because x and y are positive, this means that $x = y$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in C, there is an element x in M such that $g(x) = y$.

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1. (10 points) Let P be the set of pairs of positive integers. Suppose that $f : P \to \mathbb{R}^2$ is defined by $f(x, y) = (\frac{x}{y}, x + y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of P, i.e. pairs of positive integers. Suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(\frac{x}{y}, x + y) = (\frac{p}{q}, p + q)$. So $\frac{x}{y} = \frac{p}{q}$ and $x + y = p + q$.

   Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{py}{q}$. Substituting this into $x + y = p + q$ gives us $\frac{py}{q} + y = p + q$. So $\frac{py+yq}{q} = p + q$. I.e. $\frac{y(p+q)}{q} = p + q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

   Substituting $y = q$ into $x + y = p + q$ gives us $x + y = p + y$, so $x = p$.

   Therefore $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) $A = \{2, 3, 4, 5, 6, 7, 8 \ldots\}$, i.e. the integers ≥ 2

   $B = \{1, 2, 4, 8, 16, 32, 64, \ldots\}$, i.e. powers of 2 starting with 1

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = 2^{n-2}$

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1. (10 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is onto. Let's define $g : \mathbb{N}^2 \to \mathbb{Z}$ by $g(m, n) = (2 - n)f(m)$. Prove that g is onto.

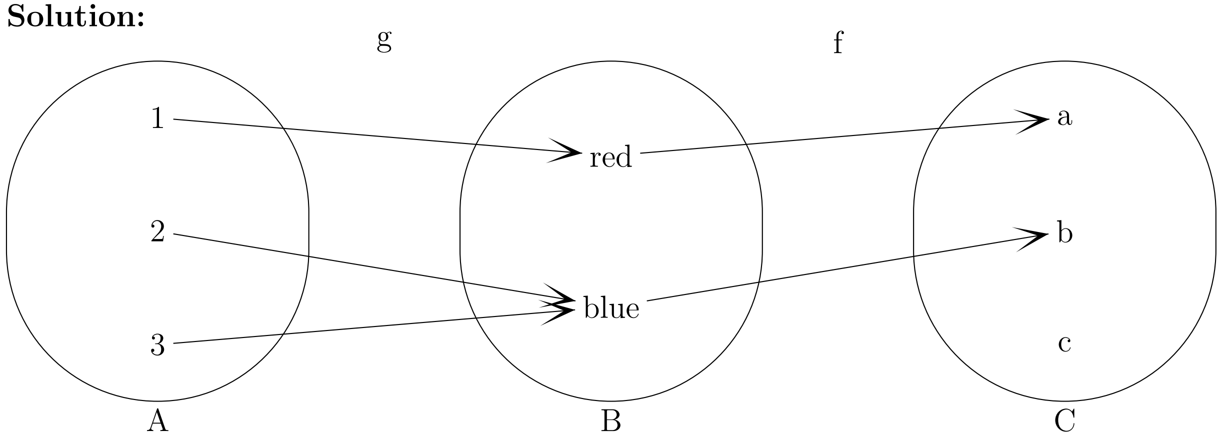
   Solution: Let y be an integer. Since f is onto, there exists $x \in \mathbb{N}$ such that $f(x) = |y|$. Then there are two cases:

   - Case 1: $y \geq 0$. Let $m = x$ and $n = 1$. Then $m, n \in \mathbb{N}$ and $g(m, n) = g(x, 1) = (2 - 1)f(x) = f(x) = |y| = y$.
   - Case 2: $y < 0$. Let $m = x$ and $n = 3$. Then $m, n \in \mathbb{N}$ and $g(m, n) = g(x, 3) = (2 - 3)f(x) = -f(x) = -|y| = -(-y) = y$.

   Since this argument works for any choice of y, we have shown that g is onto.

2. (5 points) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if g is onto, then $f \circ g$ is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

   Solution:



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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and let $f : \mathbb{N}^2 \to \mathbb{Z}$ be defined by $f(n,m) = (m-1)g(n)$. Prove that f is onto.

   Solution: Let a be an integer.

   Case 1) $a \geq 0$. Since g is onto, we can find a natural number n such that $g(n) = a$. Let $m = 2$. Then $f(n,m) = (2-1)g(n) = 1 \cdot a = a$.

   Case 2) $a \leq 0$. Then $(-a)$ is a natural number. Since g is onto, we can find a natural number n such that $g(n) = (-a)$. Let $m = 0$. Then $f(n,m) = (0-1)g(n) = (-1) \cdot (-a) = a$.

   So we've found a point (n,m) such that $g(n,m) = a$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in M, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{4x-1}{2x+5}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{4x-1}{2x+5} = \frac{4y-1}{2y+5}$.

   Multiplying by the two denominators gives us $(4x-1)(2y+5) = (4y-1)2x+5$. That is $8xy - 2y + 20x - 5 = 8xy - 2x + 20y - 5$. So $-2y + 20x = -2x + 20y$. So $22x = 22y$. And therefore $x = y$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is onto but not one-to-one. Be specific.

   Solution: Let $f(n) = n$ if $n \geq 0$, and $f(n) = n + 1$ if $n < 0$. Then f is onto. However, it's not one-to-one because 0 has two pre-images.