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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{2}{p^2+2p} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$

**Solution:** Proof by induction on $n$.

Base case(s): At $n = 1$, $\sum_{p=1}^{1} \frac{2}{p(p+2)} = \frac{2}{3} = \frac{3}{2} - \frac{1}{2} - \frac{1}{3}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$ is true for all $n = 1, 2, \ldots, k$.

Rest of the inductive step:

Notice that $\sum_{p=1}^{k+1} \frac{2}{p(p+2)} = \sum_{p=1}^{k} \frac{2}{p(p+2)} + \frac{2}{(k+1)(k+3)}$. By the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(k+1)} - \frac{1}{(k+2)}$. So we have

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{2}{p(p+2)} &= \frac{3}{2} - \frac{1}{(k+1)} - \frac{1}{(k+2)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+1)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} + \frac{2-(k+3)}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} + \frac{-(k+1)}{(k+1)(k+3)} = \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+3)},
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+3)}$, which is what we needed to prove.

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If $f$ is a function, recall that $f'$ is its derivative. Recall the product rule: if $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$. Assume we know that the derivative of $f(x) = x$ is $f'(x) = 1$.

Use (strong) induction to prove the following claim:

For any positive integer $n$, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

**Solution:** Proof by induction on $n$.

Base case(s): $n = 1$. Then $f(x) = x$. So $f'(x) = 1$. But also $nx^{n-1} = 1 \cdot n^0 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$, for $n = 1, \ldots, k$.

Rest of the inductive step: Suppose that $f(x) = x^{k+1}$. Let $g(x) = x$ and $h(x) = x^k$. By the product rule $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Since $g(x) = x$, we know that $g'(x) = 1$. By the inductive hypothesis we know that $h'(x) = kx^{k-1}$.

So $f'(x) = g'(x)h(x) + g(x)h'(x) = 1 \cdot x^k + x \cdot kx^{k-1}$. Simplifying, we get $f'(x) = x^k + kx^k = (1+k)x^k$. So $f'(x) = (1+k)x^k$, which is what we needed to show.

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Let $A$ be a constant integer. Use (strong) induction to prove the following claim. Remember that $0! = 1$.

Claim: For any integer $n \geq A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

**Solution:** Proof by induction on $n$.

Base case(s): At $n = A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{A!}{A!0!} = 1 = \frac{(A+1)!}{(A+1)!0!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$ is true for all $n = A, \ldots, k$.

Rest of the inductive step:

In particular, $\sum_{p=A}^{k} \frac{p!}{A!(p-A)!} = \frac{(k+1)!}{(A+1)!(k-A)!}$ So then

$$\begin{aligned}
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} &= \sum_{p=A}^{k} \frac{p!}{A!(p-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1)!}{(A+1)!(k-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1-A)(k+1)!}{(A+1)!(k+1-A)!} + \frac{(A+1)(k+1)!}{(A+1)!(k+1-A)!} \\
&= \frac{(k+2)(k+1)!}{(A+1)!(k+1-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}
\end{aligned}$$

So $\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for all positive integers $n$

**Solution:** Proof by induction on $n$.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} 2(-1)^p p^2 = 2(-1)^1 1^2 = -2$. And $(-1)^n n(n+1) = (-1)^1 1 \cdot 2 = -2$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for $n = 1, 2, \ldots k$.

Rest of the inductive step:

$$\sum_{p=1}^{k+1} 2(-1)^p p^2 = 2(-1)^{k+1}(k+1)^2 + \sum_{p=1}^{k} 2(-1)^p p^2$$

By the inductive hypothesis, we know that $\sum_{p=1}^{k} 2(-1)^p p^2 = (-1)^k k(k+1)$. Substituting this into the previous equation, we get

$$\begin{aligned}
\sum_{p=1}^{k+1} 2(-1)^p p^2 &= 2(-1)^{k+1}(k+1)^2 + (-1)^k k(k+1) \\
&= (k+1)(-1)^{k+1}(2(k+1) - k) \\
&= (k+1)(-1)^{k+1}(k+2) = (-1)^{k+1}(k+1)(k+2)
\end{aligned}$$

So $\sum_{p=1}^{k+1} 2(-1)^p p^2 = (-1)^{k+1}(k+1)(k+2)$ which is what we needed to show.

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Use (strong) induction and the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ to prove the following claim:

For all natural numbers $n$, $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$

**Solution:** Proof by induction on $n$.

Base case(s): At $n = 0$, $(\sum_{i=0}^{n} i)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:
Starting with the lefthand side of the equation for $n = k + 1$, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = \left((k+1) + \sum_{i=0}^{k} i\right)^2 = (k+1)^2 + 2(k+1)\sum_{i=0}^{k} i + \left(\sum_{i=0}^{k} i\right)^2$$

By the inductive hypothesis $\left(\sum_{i=0}^{k} i\right)^2 = \sum_{i=0}^{k} i^3$. Substituting this and the fact we were told to assume, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = (k+1)^2 + 2(k+1)\frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3$$

So $\left(\sum_{i=0}^{k+1} i\right)^2 = \sum_{i=0}^{k+1} i^3$ which is what we needed to show.

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The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

$\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.

**Solution:** Proof by induction on $n$.

**Base case(s):** At $n = 2$, $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = (1 - \frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

**Inductive Hypothesis** [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

**Rest of the inductive step:** In particular, from the inductive hypothesis $\prod_{p=2}^{k}(1 - \frac{1}{p^2}) = \frac{k+1}{2k}$.

So

$$\begin{aligned}
\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) &= (\prod_{p=2}^{k}(1 - \frac{1}{p^2}))(1 - \frac{1}{(k+1)^2}) \\
&= (\frac{k+1}{2k})(1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2} \\
&= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2}{2k(k+1)} - \frac{1}{2k(k+1)} \\
&= \frac{(k+1)^2 - 1)}{2k(k+1)} = \frac{k^2 + 2k)}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)}
\end{aligned}$$

So $\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.