Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Let's define two sets as follows:

$$A = \{(x, y) \in \mathbb{R}^2 : y = 3x + 7\}$$

$$B = \{\lambda(-2, 1) + (1 - \lambda)(1, 10) : \lambda \in \mathbb{R}\}$$

Prove that $A = B$ by proving two subset inclusions.

Solution:

Lemma: $\lambda(-2, 1) + (1 - \lambda)(1, 10) = (-2\lambda + (1 - \lambda), \lambda + (10 - 10\lambda)) = (1 - 3\lambda, 10 - 9\lambda)$

A ⊆ B: Let (x, y) be an element of A. Then $y = 3x + 7$ by the definition of A.

Consider $\lambda = \frac{1-x}{3}$. Then, using our lemma above, we can calculate:

$$\begin{aligned} \lambda(-2, 1) + (1 - \lambda)(1, 10) &= (1 - 3\lambda, 10 - 9\lambda) = (1 - 3\frac{1 - x}{3}, 10 - 9\frac{1 - x}{3}) \\ &= (1 - (1 - x), 10 - 3(1 - x)) = (x, 3x + 7) = (x, y) \end{aligned}$$

So we've shown that (x,y) is an element of B. Since every element of A is also an element of B, $A \subseteq B$.

B ⊆ A: Let (x,y) be an element of B. By the definition of B, we know that $(x, y) = \lambda(-2, 1) + (1 - \lambda)(1, 10)$ for some real number λ.

So, using our lemma above, $(x, y) = (1 - 3\lambda, 10 - 9\lambda)$. Then $3x + 7 = 3(1 - 3\lambda) + 7 = 3 - 9\lambda + 7 = 10 - 9\lambda = y$. So 3x+7 = y, which means that (x,y) is an element of A. Since every element of B is also an element of A, $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$, which is what we needed to show.

2. (4 points) Check the (single) box that best characterizes each item.

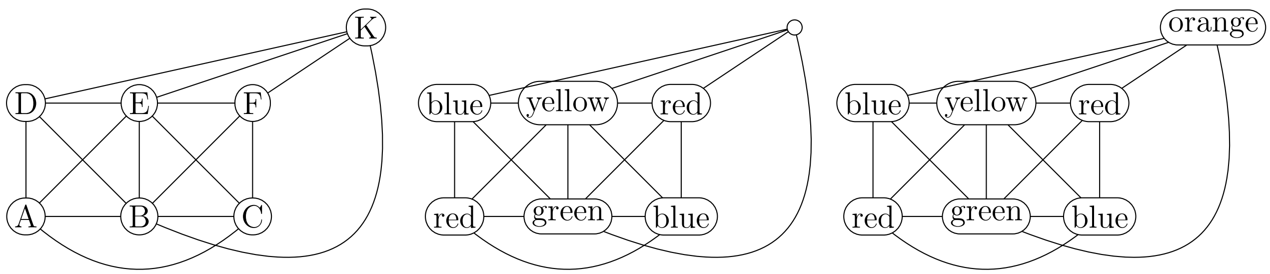
$\sum_{k=0}^{n} \frac{1}{2^k}$	$1 - (\frac{1}{2})^{n-1}$ ☐	$2 - (\frac{1}{2})^{n}$ ☑ √	$1 - (\frac{1}{2})^{n}$ ☐	$2 - (\frac{1}{2})^{n-1}$ ☐
Chromatic number of C_n.	2 ☐	3 ☐	≤ 3 ☑ √	≤ 4 ☐

Name:____________________________________

NetID:__________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.



Solution: The chromatic number is 5. The righthand picture shows that five colors are sufficient (upper bound).

To show that four colors isn't enough (lower bound), notice that A, B, C, and E form a K_4. So color these nodes with four colors as shown in the middle picture. Then F must be colored red and D must be colored blue. But then K cannot be any of the four colors.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=1}^{n} k!$	$\sum_{p=0}^{n+1} (p+1)!$ ☐	$\sum_{k=0}^{n+1} (k-1)!$ ☐	$\sum_{k=0}^{n-1} (k+1)!$ ☑	$\sum_{p=0}^{n+1} k!$ ☐
All elements of M are also elements of X.	$M = X$ ☐	$M \subseteq X$ ☑	$X \subseteq M$ ☐	
Chromatic number of G	$\mathcal{C}(G)$ ☐	$\phi(G)$ ☐	$\chi(G)$ ☑	$\parallel G \parallel$ ☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Recall that if G is a graph, then $\chi(G)$ is its chromatic number. Suppose that G is a graph and H is another graph not connected to G. Suppose G and H each have at least two nodes and at least one edge. Dr. Evil picks two adjacent nodes a and b from G, and also two adjacent nodes c and d from H. He merges G and H into a single graph T by merging b and d into a single node, and adding an edge connecting a and c. So, if G and H are as shown on the left, then T might look as shown on the right.

Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer.

Solution: $\chi(T) = \max(\chi(G), \chi(H), 3)$

The output graph contains a triangle, so it definitely requires at least three colors.

Without loss of generality, suppose that $k = \chi(G) \geq \chi(H)$. Then $\chi(T)$ must be at least k because G is a subgraph of T. Also notice that k is at least 2 because the two input graphs each contain an edge.

First, suppose k is at least 3. To color T with k colors, first color the part of T corresponding to G. We have a coloring of H that uses $\leq k$ colors, but the color choices might not be compatible with how we've started coloring T. If the two merged nodes b and d have different colors, swap the names of two colors to make them same. If a and c have the same color, swap the color of c with some third color, remembering that k is at least 3. Adjust the rest of the coloring for H to use these same choices of color names.

Special case: if $k = 2$, then we carry out the same procedure. However, we won't have any third color available to fix the color of c, so we'll have to allocate an extra color.

2. (4 points) Check the (single) box that best characterizes each item.

$\sum_{k=-2}^{n} k^2$ $\sum_{p=0}^{n+2} (p+2)^2$ ☐ $\sum_{p=0}^{n-2} (p-2)^2$ ☐ $\sum_{p=0}^{n+2} (p-2)^2$ ☑ $\sum_{p=0}^{n+2} p^2$ ☐

W_7 is a subgraph of graph H. 4 is _____ the chromatic number of H.

an upper bound on ☐ exactly ☐

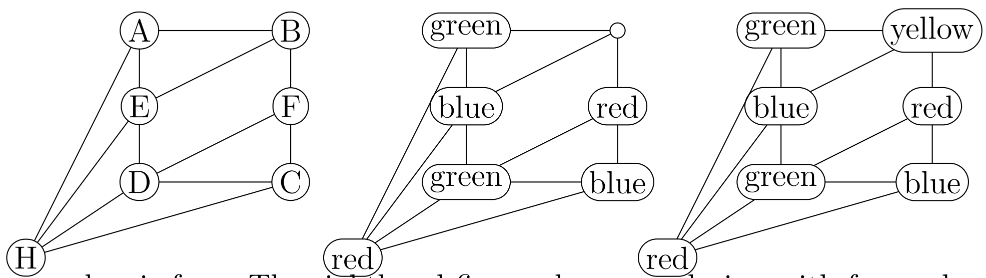
a lower bound on ☑ not a bound on ☐

Name:____

NetID:__________________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.



Solution: The chromatic number is four. The righthand figure shows a coloring with four colors (an upper bound).

To show that three colors is not enough (lower bound), first pick three colors for the triangle CDH, as shown in the middle figure. With only three colors, F must then be colored red and E blue. Then A must be colored green. But then none of the three colors will work for node B.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{i=0}^{k-1}(k \cdot i + 2)$	$\frac{k^2(k+1)}{2} + 2k$	☐	$\frac{k(k+1)}{2} + 2(k-1)$	☐
	$\frac{k^2(k-1)}{2} + 2k$	☑ √	$\frac{k(k-1)}{2} + 2(k-1)$	☐

When I poured 5 gallons of water into the bucket, some spilled over the top. 5 gallons is ____ how much the bucket holds.	an upper bound on	☑ √	exactly	☐
	a lower bound on	☐	not a bound on	☐

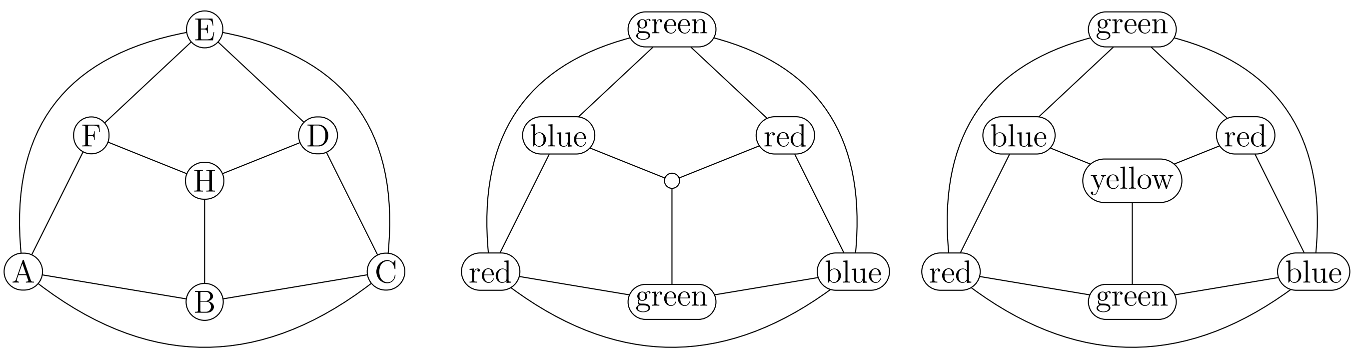
Chromatic number of a bipartite graph with at least two vertices.	1	☐	2	☐	3	☐	can't tell	☑ √

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.



Solution: The chromatic number is four. The picture on the right shows the graph colored with four colors (upper bound).

To show that three colors isn't enough, suppose that we color the triangle ABC as shown in the middle figure. Then E must be green because it's connected to A and C. But then C must be red and F blue. Then none of the three colors will work for H.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{i=1}^{p-1} i$ $\frac{p(p-1)}{2}$ [√] $\frac{(p-1)^2}{2}$ [ ] $\frac{p(p+1)}{2}$ [ ] $\frac{(p-1)(p+1)}{2}$ [ ]

I heated 2 liters of milk in my big pot. 2 liters is _____ how much the pot holds.

an upper bound on [ ] exactly [ ]
a lower bound on [√] not a bound on [ ]

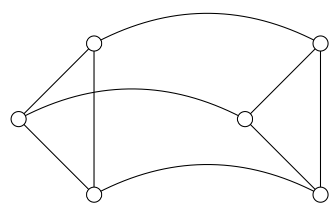
Chromatic number of a graph containing a W_n. ≥ 2 [√] ≤ 3 [ ] $\geq n$ [ ] can't tell [ ]

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (11 points) Recall that if G is a graph, then $\chi(G)$ is its chromatic number. Let's define the "doubled" version of a graph G as follows: make two copies of G and add an edge joining each pair of corresponding nodes. For example, the doubled version of C_3 looks like:



Suppose that T is the doubled version of a graph G. Describe how $\chi(T)$ is related to $\chi(G)$, justifying your answer. Your answer should handle any choice for G, not just C_3.

Solution:

$\chi(T) = \max(2, \chi(G))$.

First, let's suppose that $\chi(G) \geq 2$. If $\chi(G) = n$, then we can start coloring T by coloring one copy of G with n colors. Let's call the colors $c_1, c_2, \ldots, c_n$. Now color the second copy of G using the rule that if a node in the first copy has color c_i, then the corresponding node in the second copy has color c_{i+1} if $i + 1 \leq n$ or c_1 if $i + 1 = n$. This shows that $\chi(T) = \chi(G)$.

This construction won't work if $\chi(G)$ is 1. In this case, there aren't any edges in G. So the only edges in T connect pairs of corresponding nodes. This means that T requires two colors.

2. (4 points) Check the (single) box that best characterizes each item.

Chromatic number of W_n.	2 ☐	3 ☐	≤ 3 ☐	≤ 4 ☑

10 people can row the canoe but 11 people caused it to sink. 10 is _____ how many people the canoe can carry.	an upper bound on ☐	exactly ☑	a lower bound on ☐	not a bound on ☐