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(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Also recall that $\sin 2y = 2 \sin y \cos y$ for any real number y.

Suppose that x is a real number such that $\sin x$ is non-zero. Use (strong) induction to prove that $\prod_{p=0}^{n-1} \cos(2^p x) = \frac{\sin(2^n x)}{2^n \sin x}$, for any positive integer n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\prod_{p=0}^{n-1} \cos(2^p x) = \cos x$ and $\frac{\sin(2^n x)}{2^n \sin x} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=0}^{n-1} \cos(2^p x) = \frac{\sin(2^n x)}{2^n \sin x}$, for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $\prod_{p=0}^{k-1} \cos(2^p x) = \frac{\sin(2^k x)}{2^k \sin x}$. Using this, we can compute:

$$\begin{aligned}
\prod_{p=0}^{k} \cos(2^p x) &= (\cos(2^k x)) \prod_{p=0}^{k-1} \cos(2^p x) = (\cos(2^k x)) \frac{\sin(2^k x)}{2^k \sin x} \\
&= \frac{\cos(2^k x) \sin(2^k x)}{2^k \sin x} = \frac{2 \cos(2^k x) \sin(2^k x)}{2^{k+1} \sin x} \qquad (\text{since } \sin 2y = 2 \sin y \cos y) \\
&= \frac{\sin(2 \cdot 2^k x)}{2^{k+1} \sin x} = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}
\end{aligned}$$

So $\prod_{p=0}^{k} \cos(2^p x) = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}$. This is what we needed show, since it is the claim at $n = k + 1$.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5 \qquad f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Solution: Proof by induction on n.

Base case(s): For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.
For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.
So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:
Using the definition of f and the inductive hypothesis, we get
$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$
Simplifying the algebra,

$$\begin{aligned}
4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\
&= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\
&= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3
\end{aligned}$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

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(20 points) A "triangle-free" graph is a graph that doesn't contain any 3-cycles. Use (strong) induction to prove that a triangle-free graph with $2n$ nodes has $\leq n^2$ edges, for any positive integer n. Hint: in the inductive step, remove a pair of nodes joined by an edge. How many edges from those nodes to the rest of the graph?

Solution: Proof by induction on n.

Base case(s): At n=1. The graph has only two nodes, so it cannot have more than one edge. Since $n^2 = 1$, this means the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that any triangle-free graph with $2n$ nodes has $\leq n^2$ edges, for $n = 1, \ldots, k$.

Rest of the inductive step: Let G be a triangle-free graph with 2(k+1) nodes ($k \geq 1$). There are two cases:

Case 1: G has no edges. Since ($k \geq 1$), this means that G has $\leq (k+1)^2$ edges, which is what we needed to prove.

Case 2: G has at least one edge. Let's pick two nodes a and b that are joined by an edge. Let H be the graph we get by removing a, b, and all their edges from G. By the inductive hypothesis, H has $\leq k^2$ edges.

Consider a node v in H (i.e. a node that's not a or b). Because G is triangle-free, v cannot be joined to both of a and b. So v has either one edge to a or b, or no edges. Therefore, since H has 2k nodes, there can be no more than 2k edges from a/b to the nodes in H. We also have one extra edge: the edge joining a to b.

The total number of edges in G is the number in H, plus the nodes from a/b to a node in H, plus the edge joining a to b. That total is $\leq k^2 + 2k + 1 = (k+1)^2$. This is what we needed to prove.

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(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 1 \qquad f(1) = -5$

$f(n) = -7f(n-1) - 10f(n-2), \quad \text{for } n \geq 2$

Use (strong) induction to prove that $f(n) = (-1)^n \cdot 5^n$

Solution: Proof by induction on n.

Base case(s): $f(0) = 1 = (-1)^0 \cdot 5^0$ and $f(1) = -5 = (-1)^1 \cdot 5^1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = (-1)^n * 5^n$ for $n = 0, 1, \ldots, k-1$, for some integer $k \geq 2$.

Rest of the inductive step:

From the inductive hypothesis, we know that $f(k-1) = (-1)^{k-1} * 5^{k-1}$ and $f(k-2) = (-1)^{k-2} * 5^{k-2}$

So then we have

$$\begin{aligned}
f(k) &= -7 \cdot f(k-1) - 10 \cdot f(k-2) \\
&= -7 \cdot (-1)^{k-1} * 5^{k-1} + -10 \cdot (-1)^{k-2} * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 10 \cdot (-1)^k * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 2 \cdot (-1)^k * 5^{k-1} \\
&= 5 \cdot (-1)^k * 5^{k-1} = 5 \cdot (-1)^k * 5^k
\end{aligned}$$

So $f(k) = 5 \cdot (-1)^k * 5^{k-1} = (-1)^k * 5^k$ which is what we needed to show.

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(20 points) Use (strong) induction to prove that, for all positive integers n, $x^2 + y^2 = z^n$ has a positive integer solution. (That is, a solution in which x, y, and z are all positive integers.) Hints: (1) notice that $3^2 + 4^2 = 5^2$ and (2) use the solution for $n = k - 2$ (not $n = k - 1$) to build a solution for $n = k$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, one solution to $x^2 + y^2 = z$ is $x = 1$, $y = 2$, and $z = 5$.

At $n = 2$, one solution to $x^2 + y^2 = z^2$ is $x = 3$, $y = 4$, and $z = 5$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there is a positive integer solution to $x^2 + y^2 = z^n$ for $n = 1, 2, \ldots, k - 1$.

Rest of the inductive step: From the inductive hypothesis, we know that there is a positive integer solution to $x^2 + y^2 = z^{k-2}$. That is, we have positive integers a, b, and c, such that $a^2 + b^2 = c^{k-2}$.

Consider $x = ac$, $y = bc$ and $z = c$. ac and bc are positive integers because a, b, and c are positive integers. Then

$$x^2 + y^2 = (ac)^2 + (bc)^2 = c^2(a^2 + b^2) = c^2(c^{k-2}) = c^k$$

So $x = ac$, $y = bc$ and $z = c$ is a positive integer solution to $x^2 + y^2 = z^k$, which is what we needed to show.

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(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$g(0) = 0 \qquad g(1) = \frac{4}{3}$

$g(n) = \frac{4}{3}g(n-1) - \frac{1}{3}g(n-2), \quad \text{for } n \geq 2$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Solution: Proof by induction on n.

Base case(s): $n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$ So the claim holds.

$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$ So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ for some integer $k \geq 2$.

Inductive Step:

We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$