Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Express $f(n)$ in terms of $f(n/4^{13})$ (assuming n is large enough that this input hasn't reached the base case). Express your answer using a summation and show your work. Do **not** finish the process of finding the closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 2f(n/4) + n \\ &= 2(2f(n/4^2) + n/4) + n \\ &= 2(2(2f(n/4^3) + n/4^2) + n/4) + n \\ &= 2^3 f(n/4^3) + n/2^2) + n/2) + n \\ &= 2^k f(n/4^k) + n\sum_{p=0}^{k-1} 1/2^p \\ &= 2^{13} f(n/4^{13}) + n\sum_{p=0}^{12} 1/2^p \end{aligned}$$

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + n^2 \\ &= 4(4g(n/4) + (n/2)^2) + n^2 \\ &= 4(4(4g(n/8) + (n/4)^2) + (n/2)^2) + n^2 \\ &= 4^3 g(n/8) + n^2 + n^2 + n^2 \\ &= 4^3 g(n/8) + 3n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The Fibonacci numbers can be defined recursively by $F(0) = 0$, $F(1) = 1$, and $F(n+1) = F(n) + F(n-1)$ for all integers ...	$n \geq 0$ ☐	$n \geq 1$ ☑	$n \geq 2$ ☐

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 3f(n-1) + n^2 \\ &= 3(3f(n-2) + (n-1)^2) + n^2 \\ &= 3(3(3f(n-3) + (n-2)^2) + (n-1)^2) + n^2 \\ &= 27f(n-3) + 9(n-2)^2 + 3(n-1)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The Fibonacci numbers can be defined recursively by $F(0) = 0$, $F(1) = 1$, and $F(n+2) = F(n) + F(n+1)$ for all integers ...	$n \geq 0$ ☑	$n \geq 1$ ☐	$n \geq 2$ ☐

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= 3 \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + \sum_{p=0}^{k-1} n2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = n^2$.

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p \\ &= 4^{\log_2 n} \cdot 3 + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 3n^2 + n(2^{\log_2 n} - 1) \\ &= 3n^2 + n(n-1) = 4n^2 - n \end{aligned}$$

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$f(n) = 2^k f(n/4^k) + n \sum_{p=0}^{k-1} 1/2^p$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} f(n) &= 2^{\log_4 n} f(1) + n \sum_{p=0}^{\log_4 n - 1} 1/2^p \\ &= 0 + n(2 - \frac{1}{2^{\log_4 n - 1}}) \\ &= n(2 - \frac{2}{2^{\log_4 n}}) = n(2 - \frac{2}{\sqrt{n}}) = 2(n - \sqrt{n}) \end{aligned}$$

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (8 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Express $g(n)$ in terms of $g(n/4^3)$ (where $n \geq 64$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 2g(n/4) + n \\ &= 2(2g(n/4^2 + n/4) + n \\ &= 2(2(2g(n/4^3) + n/4^2) + n/4) + n \\ &= 2^3 g(n/4^3) + n(1/2^2 + 1/2 + 1) \\ &= 2^3 g(n/4^3) + \frac{7}{4}n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The chromatic number of the 4-dimensional hypercube Q_4 — 2 [√] 3 [ ] 4 [ ] 5 [ ]