Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} > 2^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 > 4 = 2^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} > 2^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 2^k$.

Also notice that $2k + 1 > k + 1$ because $k \geq 0$. So $\frac{2k+1}{k+1} > 1$.

Then we can compute

$$\begin{aligned} \frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\ &> \frac{(2k+2)(2k+1)}{(k+1)^2}2^k \\ &= \frac{(k+1)(2k+1)}{(k+1)^2}2^{k+1} = \frac{2k+1}{k+1}2^{k+1} > 2^{k+1} \end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} > 2^{k+1}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number x, where $0 < x < 1$, $(1-x)^n \geq 1-nx$.

Let x be a real number, where $0 < x < 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1-x)^n = (1-x)^0 = 1$ and $1 - nx = 1 + 0 = 1$. So $(1-x)^n \geq 1 - nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1-x)^n \geq 1 - nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1-x)^k \geq 1 - kx$. Notice that $(1-x)$ is positive since $0 < x < 1$. So $(1-x)^{k+1} \geq (1-x)(1-kx)$.

But $(1-x)(1-kx) = 1 - x - kx + kx^2 = 1 - (1+k)x + kx^2$.

And $1 - (1+k)x + kx^2 \geq 1 - (1+k)x$ because kx^2 is non-negative.

So $(1-x)^{k+1} \geq (1-x)(1-kx) \geq 1-(1+k)x$, and therefore $(1-x)^{k+1} \geq 1-(1+k)x$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $(2n)!^2 < (4n)!$ for all positive integers.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(2n)!^2 = (2!)^2 = 2^2 = 4$ And $(4n)! = 4! = 24$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(2n)!^2 < (4n)!$ for $n = 1, 2, \ldots, k$.

Inductive Step: At $n = k + 1$, we have

$(2(k+1))!^2 = (2k+2)!^2 = [(2k+2)(2k+1)(2k!)]^2 = (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2$

Also $(4(k+1))! = (4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$

Also notice that $(2k+2)(2k+2)(2k+1)(2k+1) < (4k+4)(4k+3)(4k+2)(4k+1)$ because each of the four terms on the left is smaller than the four terms on the right.

From the inductive hypothesis, we know that $(2k)!^2 < (4k)!$.

Putting this all together, we get

$$\begin{aligned}
(2(k+1))!^2 &= (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2 \\
&< (2k+2)(2k+2)(2k+1)(2k+1)(4k)! \\
&< (4k+4)(4k+3)(4k+2)(4k+1)(4k)! \\
&= (4(k+1))!
\end{aligned}$$

So $(2(k+1))!^2 < (4(k+1))!$, which is what we needed to prove.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{1}{p^2} > \frac{3n}{2n+1}$ for all integers $n \geq 2$

Solution:

Lemma: Suppose $k \geq 2$. Then $\frac{3k}{2k+1} - \frac{3k-3}{2k-1} = \frac{6k-3k}{4k^2-1} - \frac{6k^2-3k-3}{4k^2-1} = \frac{3}{4k^2-1}$

Notice that $3k^2 < 4k^2 - 1$, since $k \geq 2$. So $\frac{3}{4k^2-1} < \frac{1}{k^2}$.

Combining these two equations, we get $\frac{3k}{2k+1} - \frac{3k-3}{2k-1} < \frac{1}{k^2}$.

Proof by induction on n.

Base Case(s): At $n = 2$, $\sum_{p=1}^{n} \frac{1}{p^2} = 1 + \frac{1}{4} > 1 + \frac{1}{5} = \frac{6}{5} = \frac{3n}{2n+1}$ So the claim holds at $n = 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{p^2} > \frac{3n}{2n+1}$ for $n = 2, \ldots, k-1$.

Inductive Step:

By the inductive hypothesis $\sum_{p=1}^{k-1} \frac{1}{p^2} > \frac{3k-3}{2k-1}$.

By the lemma above, $\frac{1}{k^2} + \frac{3k-3}{2k-1} > \frac{3k}{2k+1}$.

So $\sum_{p=1}^{k} \frac{1}{p^2} = \frac{1}{k^2} + \sum_{p=1}^{k-1} \frac{1}{p^2} > \frac{1}{k^2} + \frac{3k-3}{2k-1} > \frac{3k}{2k+1}$.

So $\sum_{p=1}^{k} \frac{1}{p^2} > \frac{3k}{2k+1}$, which is what we needed to show.

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(15 points) Let function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be defined by

$$f(1) = f(2) = 1$$

$$f(n) = \frac{1}{2}f(n-1) + \frac{1}{f(n-2)}$$

Use (strong) induction to prove that $1 \leq f(n) \leq 2$ for all positive integers n.

Hint: prove both inequalities together using one induction.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ and $n = 2$, $f(n) = 1$. So $1 \leq f(n) \leq 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $1 \leq f(n) \leq 2$ for $n = 1, 2, \ldots k-1$.

Inductive Step: From the inductive hypothesis, we know that $1 \leq f(k-1) \leq 2$ and $1 \leq f(k-2) \leq 2$.

So $\frac{1}{2} \leq \frac{1}{2}f(k-1) \leq \frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \leq \frac{1}{f(k-2)} \leq \frac{1}{1} = 1$.

Using the upper bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \leq 1 + 1 = 2$.

Using the lower bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \geq \frac{1}{2} + \frac{1}{2} = 1$.

So $1 \leq f(k) \leq 2$, which is what we needed to show.

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Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} < 4^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 < 16 = 4^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} < 4^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 4^k$.

Then we can compute

$$\begin{aligned}
\frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\
&< \frac{(2k+2)(2k+1)}{(k+1)^2}4^k \\
&< \frac{(2k+2)(2k+2)}{(k+1)^2}4^k = \frac{4(k+1)(k+1)}{(k+1)^2}4^k \\
&= 4 \cdot 4^k = 4^{k+1}
\end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} < 4^{k+1}$, which is what we needed to show.