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(8 points) The Google interviewer suggests that $\binom{n}{k}$ can be computed very efficiently using the equation $\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$. Is this formula correct? Assume $k > 0$. Briefly justify your answer.

Solution: This formula is correct.

$$\frac{n+1-k}{k}\binom{n}{k-1} = \frac{n+1-k}{k}\frac{n!}{(k-1)!(n-(k-1))!} = \frac{(n+1-k)n!}{k(k-1)!(n-(k-1))!} = \frac{(n+1-k)n!}{k!(n-k+1)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every dog d, if d is a terrier, then d is not large and d is noisy.

Solution: There is a dog d, such that d is a terrier, but d is large or d is not noisy.

(2 points) Check the (single) box that best characterizes each item.

V is the vertex set of a tree with n edges. $\lvert\mathbb{P}(V)\rvert =$	2^{n-1} ☐	2^n ☐	not determined ☐
	2^{n+1} ☑ √	n ☐	

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(9 points) Every hacker is a black hat or a white hat (and not both). White hats always tell the truth. Black hats always lie. Alfred says to you "I am a black hat." Use proof by contradiction to show that Alfred is not a hacker.

Solution: Suppose not. That is, suppose that Alfred is a hacker. Since Alfred is a hacker, there are two possibilities.

Case 1: Alfred is a white hat. This means his statement should be true. But he said he was a black hat.

Case 2: Alfred is a black hat. This means his statement should be fals. That is, since he said he was a black hat, he should be a white hat.

In both cases, we have a contradiction: Alfred is supposedly both a white hat and a black hat.

(6 points) If $x, y, z \in \mathbb{N}$, how many solutions are there to the equation $x + y + z = 25$?

Solution: Imagine that the number 25 represents 25 objects that can be chosen in three different types: type x, type y, and type z. To indicate type we divide the objects into three bins, i.e., place 3 dividers into the list of objects. The number of ways to place the objects into the bins is the number of different solutions and is given by the expression for combinations with repetition

$$\binom{25+3-1}{25} = \binom{27}{25} = \binom{27}{2} = 351$$

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(8 points) A triomino is a triangular tile with a number on each edge. In our set of triominos, the numbers range from 0 to 5. So possible tiles include 5-3-4, 0-4-4, and 3-3-3. Tiles can be turned over: Also notice that a tile is the same if you rotate it. So 5-3-4 is the same tile as 3-4-5, 4-5-3, and also 5-4-3. How many distinct tiles are in our set?

Solution: All three sides have the same number: 6 tiles.

Two sides have the same number: 6 choices for the duplicated number, five choices for the single number. So 30 different tiles.

All three sides have different numbers: There are $\binom{6}{3} = 20$ ways to pick the three numbers. The order does not matter because we can get from any order to any other order by rotating and/or turning the tile over.

Total number of tiles is 56.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every tiger k, if k is orange, then k is large and k is not friendly.

Solution: There is a tiger k such that k is not large or k is friendly, but k is orange.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 4 flowers chosen from 17 possible varieties (zero or more of each variety).

$\binom{17}{5}$ ☐	$\binom{20}{4}$ ☑ √	$\binom{20}{3}$ ☐
$\binom{17}{4}$ ☐	$\binom{21}{4}$ ☐	$\frac{17!}{4!}$ ☐

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(9 points) Use proof by contradiction to show that, in any group of 7 people, there is at least one person who knows an even number of people. (Assume that "knowing someone" is symmetric.)

Solution: Suppose not. That is, suppose that we have a group of 7 people, in which each person knows an odd number of the other people.

Since knowing someone is symmetric, the total number of "knows" relationships must be even. (Each relationship goes both ways.) However, if we add up the number of supposed relationships, we have the sum of 7 odd numbers, which must be odd.

We have found a contradiction, so the original claim must have been correct.

(6 points) Margaret's home is defended from zombies by wallnuts, peashooters, and starfruit. She has a row of 20 pedestals on which they can stand, and she needs to use at least one starfruit. How many options does she have for the placing defenders on the pedestals?

Solution: We're making an ordered arrangement, and we have three choices for each pedestal. If we ignore the constraint about one starfruit, there are 3^{20} possible arrangements.

There are 2^{20} arrangements that feature walnuts and peashooters, but no starfruits.

So we have $3^{20} - 2^{20}$ arrangements that meet all the requirements.

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(9 points) Use proof by contradiction to show that $\sqrt{\sqrt{2}}$ is not rational. (You may use the fact that $\sqrt{2}$ is not rational.)

Solution: Suppose not. That is, suppose that $\sqrt{\sqrt{2}}$ were rational.

Since $\sqrt{\sqrt{2}}$ is rational, then $\sqrt{\sqrt{2}} = \frac{m}{n}$, where m and n are integers, $n \neq 0$. Squaring both sides, we get $\sqrt{2} = \frac{m^2}{n^2}$. Since m and n are both integers, m^2 and n^2 are both integers. So $\sqrt{2} = \frac{m^2}{n^2}$ implies that $\sqrt{2}$ is rational. But we know that $\sqrt{2}$ is not rational.

Since the negated claim led to a contradiction, the original claim must be true.

(6 points) In the town of West Fork, the streets are laid out in a uniform square grid. Alvin's school lies 6 blocks east and 9 blocks north of his house. So (since there are no diagonal roads) he travels 15 blocks to school. How many different 15-block paths can he choose from? Show your work or justify your answer.

Solution: We need to pick 6 of the 15 moves to be the ones where he walks east. That is, we have $\binom{15}{6}$ ways to form a path.

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(9 points) Use proof by contradiction to show that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $x^2 - y^2 = 10$. Factoring the lefthand side, we get $(x-y)(x+y) = 10$. $(x-y)$ and $(x+y)$ must be integers since x and y are integers.

Ignoring sign, there are only two ways to factor 10: $2 \cdot 5$ or $1 \cdot 10$. In both cases, exactly one of the factors is odd, so the sum of the two factors is odd. But the sum of $(x-y)$ and $(x+y)$ is $2x$, which is even.

We have found a contradiction, so the original claim must have been correct.

(6 points) Margaret's home is defended from zombies by wallnuts, peashooters, and starfruit. She has room to place 20 of these on her lawn. How many options does she have for her set of defenders?

Solution: This is a combinations with repetition problem. We have 20 objects and three types, thus two dividers. So the number of options is $\binom{20+2}{2} = \binom{22}{2}$.