CS 173, Fall 18 Examlet 7, Part A

Name:____

NetID:____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

Use (strong) induction to prove the following claim:

Claim: For all integers a, b, n, $n \geq 1$, if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$.

Use this definition in your proof: $x \equiv y \pmod p$ if and only if $x = y + kp$ for some integer k.

Solution:

Proof by induction on n.

Base case(s): At $n = 1$, our claim becomes "if $a \equiv b \pmod 7$ then $a \equiv b \pmod 7$" which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for all integers a, b, n, where $n = 1, \ldots, k$,

a and b need to be introduced at some point in this proof, but there's several places you might do this. For example, you could say "let a and b be integers" right at the start. Then your inductive hypothesis would just be "if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for $n = 1, \ldots, k$." We won't get picky about this when grading.

Rest of the inductive step:

Let a and b be integers.

Suppose that $a \equiv b \pmod 7$. then $a = b + 7p$ for some integer p.

From the inductive hypothesis, we know that $a^k \equiv b^k \pmod 7$, So $a^k = b^k + 7q$ for some integer q.

Combining these two equations, we get that

$$a^{k+1} = (b + 7p)(b^k + 7q) = b^{k+1} + 7(pb^k + bq + 7pq)$$

$pb^k + bq + 7pq$ is an integer since p, q, and b are integers. So we know that $a^{k+1} \equiv b^{k+1} \pmod 7$, which is what we needed to prove.

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Use (strong) induction to prove the following claim

Claim: $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, for all natural numbers n and all real numbers $p \neq 1$.

Solution: Proof by induction on n.

Base case(s): at $n = 0$, $\sum_{k=0}^{n} p^k = p^0 = 1$. And $\frac{p^{n+1}-1}{p-1} = \frac{p-1}{p-1} = 1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, all real numbers $p \neq 1$. and all natural numbers $n = 0, \ldots, j$.

p needs to be introduced somewhere, but there are several options. For example, you could say "let p be a real number $\neq 1$" before you give the inductive hypothesis. We won't be picky about this when grading.

Notice that the moving variable for the summation is k, so you can't also use k for the bound on the induction variable. You need to use a fresh variable name for one of the two.

Rest of the inductive step: Let p be a real number $\neq 1$.

Then $\sum_{k=0}^{j+1} p^k = p^{j+1} + \sum_{k=0}^{j} p^k$

By the inductive hypothesis, we know that $\sum_{k=0}^{j} p^k == \frac{p^{j+1}-1}{p-1}$. Substituting this into the previous equation, we get

$$\sum_{k=0}^{j+1} p^k = p^{j+1} + \frac{p^{j+1}-1}{p-1} = \frac{p^{j+1}(p-1) + p^{j+1} - 1}{p-1} = \frac{p^{j+2} - p^{j+1} + p^{j+1} - 1}{p-1} = \frac{p^{j+2}-1}{p-1}$$

So $\sum_{k=0}^{j+1} p^k = \frac{p^{j+2}-1}{p-1}$ which is what we needed to show.