CS 173, Fall 18 Examlet 8, Part A

Name:____

NetID:____________________ **Lecture: B**

Discussion: Friday 11 12 1 2 3 4

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 3 \qquad f(2) = 5$

$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 3$.

Use (strong) induction to prove that $f(n) = 2^n + 1$

Solution: Proof by induction on n.

Base case(s):

$n = 1$: $f(1) = 3$. Also $2^1 + 1 = 3$.

$n = 2$: $f(2) = 5$. Also $2^2 + 1 = 5$.

So the claim holds for both $n = 1$ and $n = 2$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 2^n + 1$ for $n = 1, 2, \ldots, k-1$.

Rest of the inductive step:

By the definition of f and the inducive hypothesis, we get that

$$\begin{aligned} f(k) &= 3f(k-1) = 2f(k-2) \\ &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) \end{aligned}$$

Simplifying the algebra, we get:

$$\begin{aligned} f(k) &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} - 2 \\ &= (3-1)2^{k-1} + (3-2) = 2^k + 1 \end{aligned}$$

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(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = f(1) = f(2) = 1$

$f(n) = f(n-1) + f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$. You may use the fact that $\sqrt{2}$ is smaller than 1.5.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}$. For $n = 1$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2}) = \frac{1}{\sqrt{2}}$. For $n = 2$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2}(2) = 1$ In all three cases, $f(n)$ is 1 and the value of $\frac{1}{2}(\sqrt{2})^n$ is ≤ 1.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step:

Using the definition of f and the inductive hypothesis, we get

$$f(k) = f(k-1) + f(k-3) \geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3}$$

Simplifying this expression, we get

$$\begin{aligned} f(k) &\geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3} = \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}\frac{1}{2}(\sqrt{2})^{k-1} \\ &= \frac{1}{2}(\sqrt{2})^{k-1}(1 + \frac{1}{2}) = \frac{1}{2}(\sqrt{2})^{k-1}(1.5) \\ &\geq \frac{1}{2}(\sqrt{2})^{k-1}(\sqrt{2}) = \frac{1}{2}(\sqrt{2})^k \end{aligned}$$