Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(18 points) Here is a grammar G, with start symbol S and terminal symbols a and b.

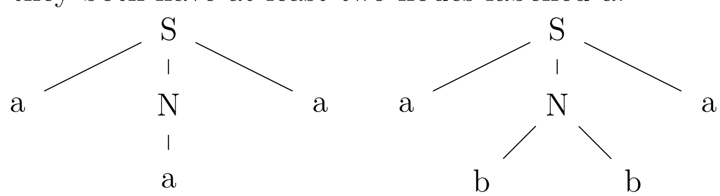
$$S \rightarrow a\,S\,a \mid S\,a\,S \mid a\,N\,a$$
$$N \rightarrow a \mid b\,b$$

Use (strong) induction to prove that any tree of height h matching (aka generated by) grammar G has at least h nodes with label a. Use $A(T)$ as shorthand for the number of a's in a tree T.

Solution:

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest trees generated by G have $h = 2$. They are as shown below and, as you can see, they both have at least two nodes labelled a.



Inductive Hypothesis [Be specific, don't just refer to "the claim"]: All trees of height h generated by G have at least h nodes labelled a, for $h = 2, 3, \ldots, k-1$. $(k \geq 3)$

Inductive Step: Suppose that T is a tree generated by G of height k. There are two cases:

Case 1: T consists of a root labelled S, with three children. The left and right children have label a. The middle child is a subtree T_1 whose root has label S. T_1 must have height $k-1$ so, by the inductive hypothesis, it contains at least $k-1$ a's. So T contains at least $(k-1)+2 = k+1$ a's.

Case 1: T consists of a root labelled S, with three children. The middle child child has label a. The left and right children are subtrees T_1 and T_2 whose roots have label S. At least one of these two subtrees has height $k-1$ so, by the indutive hypothesis, it contains at least $k-1$ a's. The middle child of T adds another a, so T must have at least k a's. (The other subtree may add additional a's.)

In either case, T has at least k nodes labelled a, which is what we needed to show.

Name:____

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Discussion: **Friday 11 12 1 2 3 4**

(18 points) Recall that a node in a full binary tree must have either 0 or 2 children. A Sleepy tree is a full binary tree in which each node is colored orange or blue, such that:

- If v is a leaf node, then v may be colored orange or blue.
- If v has two children of the same color, then v is colored blue.
- If v has two children of different colors, then v is colored orange.

Use (strong) induction to show that the root of a Sleepy tree is blue if and only if the tree has an even number of orange leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

Solution: The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): A Sleepy tree with $H = 0$ consists of a single node. If it's blue, the tree contains no orange nodes, which is even. If it's orange, the tree contains one orange node, which is odd. In both cases, the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the root of a Sleepy tree is blue if and only if the tree has an even number of orange leaves, for trees of height $h = 0, 1, \ldots, k-1$. $(k \geq 1)$

Inductive Step: Let T be a Sleepy tree of height k. There are two cases:

Case 1: the root of T is colored blue and it has two child subtrees whose roots are the same color. If both are blue, then both subtrees contain an even numbr of orange leaves by the inductive hypothesis. Similarly, if both are orange, then each contains an odd number of orange leaves. Since two odd numbers, or two even numbers, sum to an even number, T has an even number of orange leaves.

Case 2: the root of T is colored orange and it has two child subtrees whose roots are opposite colors. By the inductive hypothesis, the subtree with an orange root contain an odd number of orange leaves and the subtree with a blue root contains an even number of orange leaves. So T contains an odd number of orange leaves.

In both cases, T contains an even number of orange leaves if and only if its root is blue, which is what we needed to show.