Name:____

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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Prove the following claim, using direct proof and your best mathematical style.

For any integers m and k, if $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$, then $m^2 - 9 \leq k$.

Solution: Let m and k be integers and suppose that $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$.

Since $0 < m - 3 \leq \frac{k}{7}$, $0 < \frac{k}{7}$, so k must be positive.

Since $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$, $0 < m - 3 \leq \frac{7}{7} = 1$.

We can deduce two things from $0 < m - 3 \leq 1$. First, since k is positive, $k(m - 3) \leq k$. Second, by adding 6 to both sides, we get $m+3 \leq 7$. Since $m-3$ is positive, this implies that $(m+3)(m-3) \leq 7(m-3)$.

Combining this with $k(m - 3) \leq k$, we get $(m + 3)(m - 3) \leq 7(m - 3) \leq k$.

But $k^2 - 9 = (m + 3)(m - 3)$. So $k^2 - 9 \leq k$, which is what we needed to show.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a - b = nk$ for some integer n.

Claim: For all integers a, b, c, d, j and k (j and k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$, then $a + c \equiv b + d \pmod{j}$.

Solution:

Let a, b, c, d, j and k be integers, with j and k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a - b = nk$ for some integer n. Similarly $c \equiv d \pmod{k}$ implies that $c - d = mk$ for some integer m. By the definition of divides, $j \mid k$ implies that $k = pj$ for some integer p.

We can then calculate

$$(a + c) - (b + d) = (a - b) + (c - d) = nk + mk = (n + m)k = (n + m)pj$$

Notice that $(n + m)p$ is an integer, since n, m, and p are integers. So, by the definition of congruence mod k, $a + c \equiv b + d \pmod{j}$.

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(15 points) Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definition of "divides." ($p \nmid q$ is the negation of $p \mid q$.)

For all integers k, a, b, if $k \nmid ab$, then $k \nmid a$ and $k \nmid b$.

You must begin by explicitly stating the contrapositive of the claim.

Solution: Let's prove the contrapositive. That is, for all integers k, a, b, if $k \mid a$ or $k \mid b$, then $k \mid ab$.

So k, a, and b be integers and suppose that $k \mid a$ or k |b. There are two cases:

Case 1: $k \mid a$. Then $a = kp$ where p is an integer. Then $ab = kpb$. Let $s = pb$. Then $ab = ks$. s is an integer because p and b are integers. So $k \mid ab$.

Case 2: $k \mid b$. Then $b = kq$ where q is an integer. Then $ab = akq$. Let $t = aq$. Then $ab = kt$. t is an integer because q and a are integers. So this means that $k \mid ab$.

In both cases $k \mid ab$, which is what we needed to prove.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all real numbers p and q ($p \neq -1$), if $\frac{2}{p+1}$ and $p + q$ are rational, then q is rational.

Solution: Let p and q be real numbers, where $p \neq -1$. Suppose that $\frac{2}{p+1}$ and $p + q$ are rational.

By the definition of rational, this means that $\frac{2}{p+1} = \frac{m}{n}$ and $p+q = \frac{a}{b}$, where m, n, a, and b are integers with n and b non-zero. Notice that $\frac{2}{p+1}$ is non-zero, and therefore m is non-zero.

Since $\frac{2}{p+1} = \frac{m}{n}$, $2n = m(p + 1)$, so $p + 1 = \frac{2n}{m}$. This means that $p = \frac{2n}{m} - 1 = \frac{2n-m}{m}$.

Since $p + q = \frac{a}{b}$, $q = \frac{a}{b} - p = \frac{a}{b} - \frac{2n-m}{m} = \frac{am-b(2n-m)}{bm}$.

Since a, b, n, and m are integers, $am - b(2n - m)$ and bm are both integers. Moreover, bm must be non-zero because m and b are both non-zero. So q is the ratio of two integers, with the denominator non-zero. Therefore q is rational.

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(15 points) A natural number n is "snarky" if and only if $n = 3m + 1$, where m is a natural number. Use this definition and your best mathematical style to prove the following claim:

For all natural numbers x and y, if x and y are snarky, then $(x + y)^2$ is snarky.

Solution: Let x and y be natural numbers, and suppose that x and y are both snarky. By the definition of "snarky," this means that $x = 3m + 1$ and $y = 3p + 1$, where m and p are natural numbers.

By substitution, we then have $(x + y)^2 = (3m + 1 + 3p + 1)^2 = (3m + 3p + 2)^2$.

Let $t = m + p$. Then $(x + y)^2 = (3t + 2)^2 = 9t^2 + 12t + 4 = 3(3t^2 + 4t + 1) + 1$.

Now let $z = 3t^2 + 4t + 1$. Then $(x + y)^2 = 3z + 1$.

Notice that t is a natural number because m and p are natural numbers. And therefore z must also be a natural number. So $(x + y)^2 = 3z + 1$ means that $(x + y)^2$ is snarky, which is what we needed to prove.

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(15 points) A pair of positive integers (a, b) is defined to be a *partition* of a positive integer n if and only if $ab = n$. Using this definition and your best mathematical style, prove the following claim:

> For all positive integers a, b, and n, if (a, b) is a partition of n and $1 < a < \sqrt{n}$, then $\sqrt{n} < b < n$.

Solution: Let a, b, and n be positive integers. Suppose that (a, b) is a partition of n and $1 < a < \sqrt{n}$.

By the definition of partition, since (a, b) is a partition of n, we know that $ab = n$.

We know that $1 < a$. b was given to be positive. So $b < ab$. But $ab = n$. So $b < n$.

We know that $a < \sqrt{n}$. So $ab < b\sqrt{n}$. Since $ab = n$, this means that $n < b\sqrt{n}$. Dividing both sides by $\sqrt{n}$ gives us $\sqrt{n} < b$.

Since $b < n$ and $\sqrt{n} < b$, $\sqrt{n} < b < n$, which is what we needed to prove.