CS 173, Spring 18 Examlet 3, Part A

Name:____

NetID:__________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

For any integers s and t define $L(s,t)$ as follows:

$$L(s,t) = \{sx + ty \mid x, y \in \mathbb{Z}\}$$

Thus, $L(s,t)$ consists of all integers that can be expressed as the sum of multiples of s and t. Prove the following claim using your best mathematical style and the following definition of congruence mod k: $p \equiv q \pmod{k}$ if and only if $p = q + kn$ for some integer n.

Claim: For any integers a, b, r, where r is positive, if $a \equiv b \pmod{r}$, then $L(a,b) \subseteq L(r,b)$.

Solution: Let a, b and r be integers, where r is positive. And suppose that $a \equiv b \pmod{r}$. Then $a = b + rn$ for some integer n.

Let q be an element of $L(a,b)$. Then $q = ax + by$, where x and y are integers.

Substituting $a = b + rn$ into $q = ax + by$, we get $q = x(b + rn) + by$. So $q = (xn)r + (x + y)b$.

xn and $x + y$ are integers, because x, y, and n are integers. So $q = (xn)r + (x + y)b$ implies that $q \in L(r,b)$.

Since q was an arbitrarily chosen element of $L(a,b)$, we've shown that $L(a,b) \subseteq L(r,b)$.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(p, q) \in \mathbb{R}^2 \ | \ p = 0\}$

$B = \{(x, y) \in \mathbb{R}^2 \ | \ (x-1)^2 + y^2 = 4\}$

$C = \{(s, t) \in \mathbb{R}^2 \ | \ (s+1)^2 + t^2 = 4\}$

Prove that $B \cap C \subseteq$ A.

Solution: Let $(x, y) \in \mathbb{R}$ and suppose that $(x, y) \in B \cap C$. Then $(x, y) \in B$ and $(x, y) \in C$.

By the definitions of B and C, this means that $(x-1)^2 + y^2 = 4$ and $(x+1)^2 + y^2 = 4$. So $(x-1)^2 + y^2 = (x+1)^2 + y^2$, which means that $(x-1)^2 = (x+1)^2$. Multiplying out the two sides, we get $x^2 - 2x + 2 = x^2 + 2x + 2$ So $-2x = 2x$, so $4x = 0$, so $x = 0$.

Since $x = 0$, $(x, y) \in A$, which is what we needed to show.

[This shows more algebra steps than I'd expect for full credit.]

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$A = \{(x, y, z) \in \mathbb{Z}^3 \; : \; 2x + y = z - 1\}$

$B = \{(a, b, c) \in \mathbb{Z}^3 \; : \; 2a - b = c - 3\}$

$C = \{(p, q, r) \in \mathbb{Z}^3 \; : \; r \text{ is even}\}$

Prove that $A \cap B \subseteq C$. (Work directly from the definition of "even.")

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$. So x, y, and z are integers and $2x + y = z - 1$. Also $(x, y, z) \in B$. So $2x - y = z - 3$.

Adding together $2x+y = z-1$ and $2x-y = z-3$, we get $4x = 2z-4$. So $2x = z-2$. So $z = 2(x+1)$. Since x is an integer, $x+1$ is an integer. So $z = 2(x+1)$ implies that z is even, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(a, b) \in \mathbb{R}^2 \ : \ a = 3 - b^2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A ⊆ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{\lambda(3,2) + (1-\lambda)(5,0) \ \mid \ \lambda \in [0,1]\}$

$B = \{(x,y) \in \mathbb{R}^2 \ \mid \ x \geq y\}$

Prove that $A \subseteq B$.

Solution:

Let $(x,y) \in A$. Then $(x,y) = \lambda(3,2) + (1-\lambda)(5,0)$ for some $\lambda \in [0,1]$. So $x = 3\lambda + 5(1-\lambda) = 5 - 2\lambda$ and $y = 2\lambda$.

Since $\lambda \in [0.1]$, $\lambda \leq 1$. So $4\lambda \leq 5\lambda \leq 5$. So $5 - 4\lambda \geq 0$. And therefore $x = 5 - 2\lambda \geq 2\lambda = y$.

Since $x \geq y$, $(x,y) \in B$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(x, y, z) \in \mathbb{Z}^3 \ : \ 0 < x \leq y \leq z\}$

$B = \{(a, b, c) \in \mathbb{R}^3 \ : \ c^2 \leq a\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \ : \ p \geq 1\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so x, y, and z are integers and $0 < x \leq y \leq z$. Also $(x, y, z) \in B$, so $z^2 \leq x$.

From the first equation, we know that z is positive. Since $x \leq y \leq z$, we know that $x \leq z$. Combining this with $z^2 \leq x$, we have $z^2 \leq z$. Since z is positive, this implies that $z \leq 1$.

Notice that we now have $0 < x \leq z \leq 1$. So $0 < x \leq 1$. Since x is an integer, this means that $x = 1$. So $x \geq 1$. So $(x, y, z) \in C$, which is what we needed to show.