CS 173, Spring 18 Examlet 4, Part A

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $(xy)(p + q) < (pq)(x + y)$

Prove that T is transitive.

Solution: Let (x, y), (p, q), and (m, n) be elements of A. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $(xy)(p + q) < (pq)(x + y)$ and $(pq)(m + n) < (mn)(p + q)$

Since $m + n$ and $x + y$ are both positive, we can multiply the above equations by them to get: $(xy)(p+q)(m+n) < (pq)(x+y)(m+n)$ and $(pq)(m+n)(x+y) < (mn)(p+q)(x+y)$. Combining these two equations, we get $(xy)(p + q)(m + n) < (mn)(p + q)(x + y)$.

Since $(p + q)$ is positive, we can cancel it from both sides to get

$$(xy)(m + n) < (mn)(x + y)$$

By the definition of T, this means that $(x, y)T(m, n)$, which is what we needed to show.

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Suppose that n is some integer ≥ 2. Let's define the relation R_n on the integers such that aR_nb if and only if $a \equiv b + 1 \pmod{n}$. Prove the following claim

Claim: If R_n is symmetric, then $n = 2$.

You must work directly from the definition of congruence mod k, using the following version of the definition: $x \equiv y \pmod{k}$ iff $x - y = mk$ for some integer m. You may use the following fact about divisibility: for any non-zero integers p and q, if $p \mid q$, then $|p| \leq |q|$.

Solution: Suppose n is an integer, with $n \geq 2$. Also, suppose R_n is symmetric, where aR_nb for integers a, b iff $a \equiv b + 1 \pmod{n}$.

Suppose, then, that aR_nb for some integers a, b. Using the above definition of congruence mod k, $a - b - 1 = mn$ for some integer m. Because R_n is symmetric, bR_na, so $b - a - 1 = jn$ for some integer j. So $b = jn + a + 1$. Substituting this into $a - b - 1 = mn$, we get $a - a - jn - 2 = mn$. So $-2 = jn + mn$, so $2 = (-j - m)n$. Therefore, $n|2$ by definition of divides, since j and m are integers. Using the above divisibility fact, $|n| \leq |2|$. But we know that $n \geq 2$. So $n = 2$, which is what we needed to prove.

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Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + y = 10\}$. Consider the relation T on A defined by

$(a, b)T(p, q)$ if and only if $aq \geq bp$

Prove that T is antisymmetric.

Solution: Let (a, b) and (p, q) be points in A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(a, b)$.

By the definition of T, $(a, b)T(p, q)$ and $(p, q)T(a, b)$ imply that $aq \geq bp$ and $bp \geq aq$. So $aq = bp$.

Since (a, b) and (p, q) are in A, we know that $a + b = 10$ and $p + q = 10$. So $b = 10 - a$ and $q = 10 - p$. Substituting these equations into $aq = bp$, we get $a(10 - p) = (10 - a)p$. So $10a - ap = 10p - ap$. So $10a = 10p$. So $a = p$. But then $b = 10 - a = 10 - p = q$.

Since $a = p$ and $b = q$, $(a, b) = (p, q)$, which is what we needed to prove.

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Let T be the relation defined on $\mathbb{N}^2$ by

$$(x, y)T(p, q) \text{ if and only if } x < p \text{ or } (x = p \text{ and } y \leq q)$$

Prove that T is transitive.

Solution:

Let (x, y), (p, q) and (m, n) be pairs of natural numbers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, $(x, y)T(p, q)$ means that $x < p$ or $(x = p$ and $y \leq q)$. Similarly $(p, q)T(m, n)$ implies that $p < m$ or $(p = m$ and $q \leq n)$.

There are four cases:

Case 1: $x < p$ and $p < m$. Then $x < m$.

Case 2: $x < p$ and $p = m$. Then $x < m$.

Case 3: $x = p$ and $p < m$. Then $x < m$.

Case 4: $x = p$ and $p = m$. In this case, we must also have $y \leq q$ and $q \leq n$. So $x = m$ and $y \leq n$.

In all four cases, $(x, y)T(m, n)$, which is what we needed to show.

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Let's define a relation T between natural numbers follows:

aTb if and only if $a = b + 2k$, where k is a natural number

Working directly from this definition, prove that T is antisymmetric.

Solution: Let a and b be natural numbers and suppose that aTb and bTa.

By the definition of T, this means that $a = b+2k$ and $b = a+2j$, where k and j are natural numbers.

Substituting one equation into the other, we get $a = (a + 2j) + 2k = a + 2(j + k)$. So $2(j + k) = 0$. So $j + k = 0$.

Notice that j and k are both non-negative. So $j + k = 0$ implies that $j = k = 0$.

So $a = b$, which is what we needed to show.

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Suppose that T is a relation on the integers which is antisymmetric. Let's define a relation R on pairs of integers such that $(p, q)R(a, b)$ if and only if $(a + b)T(p + q)$ and bTq. Prove that R is antisymmetric.

Solution: Let (a, b) and (p, q) be pairs of integers. Suppose that $(a, b)R(p, q)$ and $(p, q)R(a, b)$.

By the definition of R, this means that $(a, b)R(p, q)$ means that $(p+q)T(a+b)$ and qTb. Similarly,$(p, q)R(a, b)$ means that $(a + b)T(p + q)$ and bTq.

Because T is antisymmetric, qTb and bTq implies that $q = b$. Similarly, $(p + q)T(a + b)$ and $(a + b)T(p + q)$ implies that $p + q = a + b$.

Since $q = b$ and $p + q = a + b$, $p = a$. So $(p, q) = (a, b)$, which is what we needed to prove.