Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) + y, y + 3)$. Prove that g is onto.

Solution: Suppose that (a, b) is a pair of integers.

Consider $c = a - b + 3$. c is an integer, since a and b are integers. Since f is onto, this means there is an integer x such that $f(x) = c$.

Now, let $y = b - 3$. We can then calculate:

$$g(x, y) = (f(x) + y, y + 3) = (c + y, (b - 3) + 3) = ((a - b + 3) + (b - 3), b) = (a, b)$$

So we've found a point (x, y) such that $g(x, y) = (a, b)$, which is what we needed to show.

CS 173, Spring 18 Examlet 6, Part A

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : (0, \infty) \to (\frac{5}{4}, \infty)$ is defined by $f(x) = \frac{5x^2+3}{4x^2}$. Proof that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution: Let x and y be positive real numbers and suppose that $f(x) = f(y)$. By the definition of f, this translates into

$\frac{5x^2+3}{4x^2} = \frac{5y^2+3}{4y^2}$

So $\frac{5}{4} + \frac{3}{4}\frac{1}{x^2} = \frac{5}{4} + \frac{3}{4}\frac{1}{y^2}$

So $\frac{3}{4}\frac{1}{x^2} = \frac{3}{4}\frac{1}{y^2}$

So $\frac{1}{x^2} = \frac{1}{y^2}$

So $x^2 = y^2$.

Since x and y are known to be positive, this implies that $x = y$, which is what we needed to show.

CS 173, Spring 18 Examlet 6, Part A

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = xy + yx^2 - x^2$. Prove that f is onto.

Solution:

Notice that $f(x, y) = xy + (y - 1)x^2$.

Let p be an integer. We need to find a pre-image for p.

Consider $m = (p, 1)$.

m is an element of $\mathbb{Z}^2$. We can compute

$$f(m) = p \cdot 1 + (1 - 1)p^2 = p + 0 \cdot p^2 = p$$

So m is a pre-image of p.

Since we can find a pre-image for an arbitrarily chosen integer, f is onto.

CS 173, Spring 18 Examlet 6, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that g is one-to-one.

Solution: Let m and n be elements of A. Suppose that $g(m) = g(n)$.

Since $g(m) = g(n)$, $f(g(m)) = f(g(n))$ by substitution. Since $f(g(x)) = x$ for every $x \in A$, $f(g(m)) = m$ and $f(g(n)) = n$. So $f(g(m)) = f(g(n))$ implies that $m = n$.

Since $g(m) = g(n)$ implies that $m = n$ for any m and n in A, g is one-to-one, which is what we needed to prove.

CS 173, Spring 18 Examlet 6, Part A

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose that $f : [0, \frac{1}{2}] \rightarrow [1, \frac{5}{2}]$ is defined by $f(x) = \frac{x^2+1}{1-2x^2}$ Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution:

Let x and y be any numbers in $[0, \frac{1}{2}]$ and suppose $f(x) = f(y)$, that is

$$
\begin{aligned}
& \frac{x^2+1}{1-2x^2} = \frac{y^2+1}{1-2y^2} \\
\Rightarrow \quad & (x^2+1)(1-2y^2) = (y^2+1)(1-2x^2) \\
\Rightarrow \quad & x^2 + 1 - 2x^2y^2 - 2y^2 = y^2 + 1 - 2x^2y^2 - 2x^2 \\
\Rightarrow \quad & 3x^2 = 3y^2 \\
\Rightarrow \quad & x = y
\end{aligned}
$$

(The last step works because x and y are both positive.)

Therefore f is one-to-one.