CS 173, Spring 18 Examlet 7, Part A

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for all natural numbers n.

Recall that 0! is defined to be 1.

Solution: Proof by induction on n.

Base case(s):

At $n = 0$, $\sum_{p=0}^{n}(p \cdot p!) = 0 \cdot 0! = 0$ Also $(n+1)! - 1 = 1! - 1 = 1 - 1 = 0$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

By the inductive hypothesis $\sum_{p=0}^{k}(p \cdot p!) = (k+1)! - 1$. So

$$\begin{aligned}
\sum_{p=0}^{k+1}(p \cdot p!) &= ((n+1) \cdot (n+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= ((k+1) \cdot (k+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= (n+1) \cdot (k+1)! + (k+1)! - 1 \\
&= (k+1) \cdot (k+1)! + (k+1)! - 1 \\
&= [(k+1) + 1] \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 = (k+2)! - 1
\end{aligned}$$

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$ for all integers $n \geq 2$.

Solution:

Proof by induction on n.

Base case(s): $n = 2$. At $n = 2$, $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{1}{2}$. Also $\frac{n-1}{n} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$. for $n = 2, \ldots, k$ for some integer $k \geq 2$.

Rest of the inductive step:

Consider $\sum_{j=2}^{k+1} \frac{1}{j(j-1)}$.

By removing the top term of the summation and then applying the inductive hypothesis, we get $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \sum_{j=2}^{k} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \frac{k-1}{k}$.

Adding the two fractions together:

$\frac{1}{(k+1)k} + \frac{k-1}{k} = \frac{1}{(k+1)k} + \frac{(k+1)(k-1)}{(k+1)k} = \frac{1}{(k+1)k} + \frac{k^2-1}{(k+1)k} = \frac{k^2}{(k+1)k} = \frac{k}{(k+1)}$

So $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{k}{(k+1)}$ which is what we needed to show.

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Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $(4n)!$ is divisible by 8^n, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, the claim amounts to "4! is divisible by 8." $4! = 24$ which is clearly divisible by 8.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(4n)!$ is divisible by 8^n, for $n = 1, 2, \dots, k$.

Rest of the inductive step: At $n = k + 1$, $(4n)! = (4(k + 1))! = (4k + 4)! = (4k + 4)(4k + 3)(4k + 2)(4k + 1)(4k)!$

Now, $(4k + 4)$ is divisible by 4, and $(4k + 2)$ is divisible by 2. So $(4k + 4)(4k + 3)(4k + 2)(4k + 1)$ is divisible by 8. By the inductive hypothesis, we know that $(4k)!$ is divisible by 8^k. Combining these two facts, $(4(k + 1))!$ is divisible by 8^{k+1}, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} j(j+1) = 1(1+1) = 2$ Also, $\frac{n(n+1)(n+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$. So the two sides of the equation are equal at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for $n = 1, \ldots, k$, for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} j(j+1)$. By removing the top term of the summation and applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} j(j+1) = (k+1)(k+2) + \sum_{j=1}^{k} j(j+1) = (k+1)(k+2) + \frac{k(k+1)(k+2)}{3}$$

Simplifying the algebra:

$$(k+1)(k+2) + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)}{3} + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)\ +\ k(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$$

So $\sum_{j=1}^{k+1} j(j+1) = \frac{(k+1)(k+2)(k+3)}{3}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all positive integers n, $\sum_{p=1}^{n} p2^p = (n-1)2^{n+1} + 2$.

Solution: Proof by induction on n.

Base case(s): $n = 1$. Then $\sum_{p=1}^{n} p2^p = 1 \cdot 2^1 = 2$ and $(n-1)2^{n+1} + 2 = 0 \cdot 2^2 + 2 = 2$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} p2^p = (n-1)2^{n+1} + 2$ for $n = 1, \ldots, k$.

Rest of the inductive step:

From the inductive hypothesis $\sum_{p=1}^{k} p2^p = (k-1)2^{k+1} + 2$.

Then

$$\begin{aligned}
\sum_{p=1}^{k+1} p2^p &= (\sum_{p=1}^{k} p2^p) + (k+1)2^{k+1} \\
&= ((k-1)2^{k+1} + 2) + (k+1)2^{k+1} \\
&= ((k-1) + (k+1))2^{k+1} + 2 = 2k2^{k+1} + 2 = k2^{k+2} + 2
\end{aligned}$$

So $\sum_{p=1}^{k+1} p2^p = k2^{k+2} + 2$, which is what we needed to show.

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Use (strong) induction to prove the following claim.

Claim: For any positive integer n, 2^{4n-1} ends in the digit 8. (I.e. when written out in base-10, the one's digit is 8.)

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $2^{4n-1} = 2^3 = 8$, which ends in the digit 8.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that 2^{4n-1} ends in the digit 8, for $n = 1, \ldots, k$.

Rest of the inductive step:

In particular, 2^{4k-1} ends in the digit 8. That is $2^{4k-1} = 10p + 8$, where p is an integer. Then

$$\begin{aligned} 2^{4(k+1)-1} &= 2^{4k+4-1} = 2^{(4k-1)+4} = 2^{4k-1} \cdot 2^4 \\ &= (10p+8) \cdot 2^4 = (10p+8) \cdot 16 \\ &= 10(16p) + 8 \cdot 16 = 10(16p) + 128 \\ &= 10(16p) + 120 + 8 = 10(16p+12) + 8 \end{aligned}$$

$16p + 12$ is an integer, since p is an integer. So $2^{4(k+1)-1} = 10(16p + 12) + 8$ has a remainder of 8 when divided by 10. That is, its one's digit is 8, which is what we needed to prove.